CHAPTER 4  Personal Finance

The following is an article from a Marlboro, Massachusetts newspaper.

NEWSPAPER ARTICLE 4.1: LET’S TEACH FINANCIAL LITERACY  
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WED JAN 16, 2008

Boston - Last week legislation I filed requiring the Massachusetts Department of Education to include all aspects of personal finance as a major component in the existing math curriculum was favorably reported out by the Education Committee.

Students need to learn personal finance math skills that will help them to succeed in life. This effort is intended to highlight the need for a comprehensive offering in the study of personal finance, with the understanding that the need for such knowledge cuts across all socio-economic segments of the population.

Consider that saving levels in America are at their lowest since the Great Depression. Americans have accumulated $505 billion in credit card debt, and 81 percent of college freshman have credit cards. Additionally, if we look at the current foreclosure crisis, we know that many of these problems could have been avoided if the public had a better understanding of credit and personal finance.

Our students are graduating from high school without knowing what debt can do to them or what compound interest can do for them. We need to teach our students the fundamentals of financial literacy.

There are immediate, tangible benefits for students who are introduced to matters of personal finance as a component of their High School mathematics curriculum. It is very easy for students to pose the question, When am I ever going to use this? about derivatives and quadratic equations but, with regard to budgeting, debt, and personal investing, those practical applications are instantly apparent.

This article makes clear that borrowing and saving money are a big part of our lives and that understanding these transactions can make a huge difference to us. In
this chapter we explore basic financial terminology and mechanisms. In Section 4.1 we examine the basics of compound interest and savings, and in Section 4.2 we look at borrowing. In Section 4.3 we consider long-term savings plans such as retirement funds. In Section 4.4 we focus on credit cards, and in Section 4.5 we discuss financial terms heard in the daily news.

QUICK REVIEW: LINEAR FUNCTIONS
Linear functions play an important role in the mathematics of finance. We recall their basic properties here. For additional information see Chapter 3.

Linear functions: A linear function has a constant growth rate, and its graph is a straight line. The growth rate of the function is also referred to as the slope.

We find a formula for a linear function of \( t \) using

\[
\text{Linear function} = \text{Growth rate} \times t + \text{Initial value}.
\]

Example: If we initially have $1000 in an account and add $100 each year, the balance is a linear function because it is growing by a constant amount each year. After \( t \) years, the balance is

\[
\text{Balance after } t \text{ years} = 100t + 1000.
\]

The graph of this function is shown in Figure 4.1. Note that the graph is a straight line.

Figure 4.1: The graph of a linear function is a straight line
QUICK REVIEW: EXPONENTIAL FUNCTIONS

Exponential functions play an important role in the mathematics of finance. We recall their basic properties here. For additional information see Chapter 3.

**Exponential functions**: An increasing exponential function exhibits a constant percentage growth rate. If $r$ is this percentage growth rate expressed as a decimal, the base of the exponential function is $1 + r$. We find a formula for an exponential function of $t$ using

$$\text{Exponential function} = \text{Initial value} \times (1 + r)^t.$$  

*Example*: If we initially have $1000 in an account that grows by 10% per year, the balance is exponential because it is growing at a constant percentage rate. Now 10% as a decimal is $r = 0.10$, so $1 + r = 1.10$. After $t$ years, the balance is

$$\text{Balance after } t \text{ years} = 1000 \times 1.10^t.$$  

The graph of this function is shown in Figure 4.2. The increasing growth rate is typical of increasing exponential functions.

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**Figure 4.2**: The graph of an exponential function (with base greater than 1) gets steeper as we move to the right.
4.1 SAVING MONEY: THE POWER OF COMPOUNDING

Investments such as savings accounts earn interest over time. In this section we see how to measure the growth of such accounts. Interest can be earned in different ways, and that affects the growth in value of an investment.

TAKE AWAY FROM THIS SECTION

Understand compound interest and the difference between APR (annual percentage rate) and APY (annual percentage yield).

Simple interest

The easiest type of interest to understand and calculate is simple interest.

KEY CONCEPT

The initial balance of an account is the principal. Simple interest is calculated by applying the interest rate to the principal only, not to interest earned.

Suppose, for example, that we invest $1000 in an account that earns simple interest at a rate of 10% per year. Then we earn 10% of $1000 or $100 in interest each year. If we hold the money for 6 years, we get $100 in interest each year for 6 years. That comes to $600 interest. If we hold it for only 6 months, we get a half-year’s interest or $50.

Here is the formula for computing simple interest.

\[
\text{Simple interest earned} = \text{Principal} \times \text{Yearly interest rate (as a decimal)} \times \text{Time in years.}
\]

EXAMPLE 4.1 Calculating simple interest: an account

We invest $2000 in an account that pays simple interest of 4% each year. Find the interest earned after 5 years.

Solution:

The interest rate of 4% written as a decimal is 0.04. The principal is $2000, and the
time is 5 years. We find the interest by using these values in the simple interest formula (Formula 4.1):

\[
\text{Simple interest earned} = \text{Principal} \times \text{Yearly interest rate} \times \text{Time in years}
\]

\[
= 2000 \times 0.04/ \text{year} \times 5 \text{ years} = 400.
\]

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**TRY IT YOURSELF 4.1**

We invest $3000 in an account that pays simple interest of 3% each year. Find the interest earned after 6 years.

The answer is provided at the end of this section.

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**Compound interest**

Situations involving simple interest are fairly rare and usually occur when money is lent or borrowed for a short period of time. More often, interest payments are made in periodic installments during the life of the investment. The interest payments are credited to the account periodically, and future interest is earned not only on the original principal but also on the interest earned to date. This type of interest calculation is referred to as **compounding**.

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**KEY CONCEPT**

**Compound interest** is paid on the principal and on the interest that the account has already earned. In short, compound interest includes *interest on the interest*.

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To see how compound interest works, let’s return to the $1000 investment earning 10% per year we looked at earlier, but this time let’s assume the interest is compounded annually (at the end of each year). At the end of the first year we earn 10% of $1000 or $100—the same as with simple interest. When interest is compounded, we add this amount to the balance, giving a new balance of $1100. At the end of the second year we earn 10% interest on the $1100 balance:

\[
\text{Second year’s interest} = 0.10 \times 1100 = 110.
\]
This amount is added to the balance, so after 2 years the balance is

\[
\text{Balance after 2 years} = \$1100 + \$110 = \$1210.
\]

For comparison, we can use the simple interest formula to find out the simple interest earned after 2 years:

\[
\text{Simple interest after 2 years} = \text{Principal} \times \text{Yearly interest rate} \times \text{Time in years} = \$1000 \times 0.10/\text{year} \times 2 \text{ years} = \$200.
\]

The interest earned is $200, so the balance of the account is $1200.

After 2 years, the balance of the account earning simple interest is only $1200, but the balance of the account earning compound interest is $1210. Compound interest is always more than simple interest, and this observation suggests a rule of thumb for estimating the interest earned.

**RULE OF THUMB 4.1: ESTIMATING INTEREST**

Interest earned on an account with compounding is always at least as much as that earned from simple interest. If the money is invested for a short time, simple interest can be used as a rough estimate.

The following table compares simple interest and annual compounding over various periods. It uses $1000 for the principal and 10% for the annual rate. This table shows why compounding is so important for long-term savings.

<table>
<thead>
<tr>
<th>End of year</th>
<th>Simple interest</th>
<th>Yearly compounding</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Interest</td>
<td>Balance</td>
</tr>
<tr>
<td>1</td>
<td>10% of $1000 = $100</td>
<td>$1100</td>
</tr>
<tr>
<td>2</td>
<td>10% of $1000 = $100</td>
<td>$1200</td>
</tr>
<tr>
<td>3</td>
<td>10% of $1000 = $100</td>
<td>$1300</td>
</tr>
<tr>
<td>10</td>
<td>$100</td>
<td>$2000</td>
</tr>
<tr>
<td>50</td>
<td>$100</td>
<td>$6000</td>
</tr>
</tbody>
</table>

To understand better the comparison between simple and compound interest, observe that for simple interest the balance is growing by the same amount, $100, each year. This means that the balance for simple interest is showing linear growth. For compound interest the balance is growing by the same percent, 10%, each year. This means that the balance for compound interest is growing exponentially. The graphs
of the account balances are shown in Figure 4.3. The widening gap between the two graphs shows the power of compounding.

**Figure 4.3: Balance for simple interest is linear, and balance for compound interest is exponential**

![Graph showing simple and compound interest over time.](image)

**EXAMPLE 4.2 Calculating compound interest: annual compounding**

You invest $500 in an account that pays 6% compounded annually. What is the account balance after 2 years?

**Solution:**

Now 6% expressed as a decimal is 0.06. The first year’s interest is 6% of $500:

\[
\text{First year’s interest} = 0.06 \times 500 = \$30.00.
\]

This interest is added to the principal to give an account balance at the end of the first year of $530.00. We use this figure to calculate the second year’s interest:

\[
\text{Second year’s interest} = 0.06 \times 530.00 = \$31.80.
\]

We add this to the balance to find the balance at the end of 2 years:

\[
\text{Balance after 2 years} = 530.00 + 31.80 = \$561.80.
\]

**TRY IT YOURSELF 4.2**

Find the balance of this account after 4 years.

*The answer is provided at the end of this section.*
Other compounding periods and the APR

Interest may be compounded more frequently than once a year. For example, compounding may occur semi-annually, in which case the compounding period is half a year. Compounding may also be done quarterly, monthly, or even daily. To calculate the interest earned we need to know the period interest rate.

**KEY CONCEPT**

The period interest rate is the interest rate for a given compounding period (for example, a month).

Financial institutions report the annual percentage rate or APR. To calculate this rate they multiply the period interest rate by the number of periods in a year. This leads to an important formula.

**Formula (4.2): APR formula**

\[
\text{Period interest rate} = \frac{\text{APR}}{\text{Number of periods in a year}}.
\]

Suppose for example that we invest $500 in a savings account that has an APR of 6% and compounds interest monthly. Then there are 12 compounding periods each year. We find the monthly interest rate using

\[
\text{Monthly interest rate} = \frac{\text{APR}}{12} = \frac{6\%}{12} = 0.5\%.
\]

Each month we add 0.5% interest to the current balance. The following table shows how the account balance grows over the first few months.

<table>
<thead>
<tr>
<th>End of Month</th>
<th>Interest earned</th>
<th>New balance</th>
<th>Percent increase</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5% of $500.00=$2.50</td>
<td>$502.50</td>
<td>0.5%</td>
</tr>
<tr>
<td>2</td>
<td>0.5% of $502.50=$2.51</td>
<td>$505.01</td>
<td>0.5%</td>
</tr>
<tr>
<td>3</td>
<td>0.5% of $505.01=$2.53</td>
<td>$507.54</td>
<td>0.5%</td>
</tr>
<tr>
<td>4</td>
<td>0.5% of $507.54=$2.54</td>
<td>$510.08</td>
<td>0.5%</td>
</tr>
</tbody>
</table>
Compound interest formula

So far we have calculated the interest by hand to see how the balance grows due to compounding. Now we simplify the process by giving a formula for the balance.

If \( r \) is the period interest rate expressed as a decimal, we find the balance after \( t \) periods using:

\[
\text{Balance after } t \text{ periods} = \text{Principal} \times (1 + r)^t.
\]

Here is an explanation for the formula: Over each compounding period the balance grows by the same percentage, so the balance is an exponential function of \( t \). That percentage growth is \( r \) as a decimal, and the initial value is the principal. Using these values in the standard exponential formula

\[
\text{Exponential function} = \text{Initial value} \times (1 + r)^t
\]
gives the compound interest formula.

Let’s find the formula for the balance if $500 is invested in a savings account that pays an APR of 6% compounded monthly. The APR as a decimal is 0.06, so in decimal form the monthly rate is \( r = 0.06/12 = 0.005 \). Thus, \( 1 + r = 1.005 \). By Formula 4.3,

\[
\text{Balance after } t \text{ months} = \text{Principal} \times (1 + r)^t
\]

\[
= \$500 \times 1.005^t.
\]

We can use this formula to find the balance of the account after five years. Five years is 60 months, so we use \( t = 60 \) in the formula:

\[
\text{Balance after 60 months} = \$500 \times 1.005^{60} = \$674.43.
\]

The APR by itself does not determine how much interest an account earns. The number of compounding periods also plays a role, as the next example illustrates.

EXAMPLE 4.3 Calculating values with varying compounding periods: value of a CD

Suppose we invest $10,000 in a 5-year certificate of deposit (CD) that pays an APR of 6%.
a. What is the value of the mature CD if interest is compounded annually? (Maturity refers to the end of the life of a CD. In this case maturity occurs at 5 years.)

b. What is the value of the mature CD if interest is compounded quarterly?

c. What is the value of the mature CD if interest is compounded monthly?

d. What is the value of the mature CD if interest is compounded daily?

e. Compare your answers from parts a, b, c, and d.

Solution:

a. The annual compounding rate is the same as the APR. Now 6% as a decimal is \( r = 0.06 \). We use \( 1 + r = 1.06 \) and \( t = 5 \) years in the compound interest formula (Formula 4.3):

\[
\text{Balance after 5 years} = \text{Principal} \times (1 + r)^t = 10,000 \times 1.06^5 = 13,382.63.
\]

b. Again we use the compound interest formula. To find the quarterly rate we divide the APR by 4. The APR as a decimal is 0.06, so as a decimal the quarterly rate is

\[
r = \text{Quarterly rate} = \frac{\text{APR}}{4} = \frac{0.06}{4} = 0.015.
\]

Thus, \( 1 + r = 1.015 \). Also 5 years is 20 quarters, so we use \( t = 20 \) in the compound interest formula:

\[
\text{Balance after 20 quarters} = \text{Principal} \times (1 + r)^t = 10,000 \times 1.015^{20} = 13,468.50.
\]

c. This time we want the monthly rate, so we divide the APR by 12:

\[
r = \text{Monthly rate} = \frac{\text{APR}}{12} = \frac{0.06}{12} = 0.005.
\]

Also, 5 years is 60 months, so

\[
\text{Balance after 60 months} = \text{Principal} \times (1 + r)^t = 10,000 \times 1.005^{60} = 13,488.50.
\]
d. We assume that there are 365 days in each year, so as a decimal the daily rate is

\[ r = \text{Daily rate} = \frac{\text{APR}}{365} = \frac{0.06}{365} . \]

This is \( r = 0.00016 \) to five decimal places, but for better accuracy we won’t round this number. (See Calculation Tip 4.1 below.) Five years is \( 5 \times 365 = 1825 \) days, so

\[
\text{Balance after 1825 days} = \text{Principal} \times (1 + r)^t = $10,000 \times \left(1 + \frac{0.06}{365}\right)^{1825} = $13,498.26.
\]

e. We summarize the results above in the following table.

<table>
<thead>
<tr>
<th>Compounding period</th>
<th>Balance at maturity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yearly</td>
<td>$13,382.26</td>
</tr>
<tr>
<td>Quarterly</td>
<td>$13,468.55</td>
</tr>
<tr>
<td>Monthly</td>
<td>$13,488.50</td>
</tr>
<tr>
<td>Daily</td>
<td>$13,498.26</td>
</tr>
</tbody>
</table>

This table shows that increasing the number of compounding periods increases the interest earned even though the APR and the number of years stay the same.

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**Calculation Tip 4.1: Rounding**

Some financial calculations are very sensitive to rounding. In order to obtain accurate answers, when you use a calculator it is better to keep all the decimal places rather than to enter parts of the formula that you have rounded. You can do this by either entering the complete formula or using the memory key on your calculator to store numbers with lots of decimal places.

For instance, in part d of the example above we found the balance after 1825 days to be $13,498.26. But if we round the daily rate to 0.00016 we get $10,000 \times 1.00016^{1825} = $13,390.72. Rounding significantly affects the accuracy of the answer.

More information on rounding is given in Appendix 3.
**SUMMARY 4.1: COMPOUND INTEREST**

1. With compounding, interest is earned each period on both the principal and whatever interest has already accrued.

2. Financial institutions report the annual percentage rate (APR).

3. If interest is compounded \( n \) times per year, to find the period interest rate we divide the APR by \( n \):

   \[
   \text{Period rate} = \frac{\text{APR}}{n}.
   \]

4. We can calculate the account balance after \( t \) periods using the compound interest formula (Formula 4.3):

   \[
   \text{Balance after } t \text{ periods} = \text{Principal} \times (1 + r)^t.
   \]

   Here \( r \) is the period interest rate expressed as a decimal. Many financial formulas, including this one, are sensitive to round-off error, so it is best to do all the calculations and then round.

**APR versus APY**

Suppose we see one savings account offering an APR of 4.32% compounded quarterly and another one offering 4.27% compounded monthly. What we really want to know is which one pays us more money at the end of the year. This is what the *annual percentage yield* or APY tells us.

**KEY CONCEPT**

The *annual percentage yield* or APY is the actual percentage return earned in a year. Unlike the APR, the APY tells us the actual percentage growth per year, including returns on investment due to compounding.

A federal law passed in 1991 requires banks to disclose the APY.

To understand what the APY means, let’s look at a simple example. Suppose we invest $100 in an account that pays 10% APR compounded semi-annually. We want to see how much interest is earned in a year. The period interest rate is

\[
\frac{\text{APR}}{2} = \frac{10\%}{2} = 5\%.
\]
Over the first six months we earn 5% of $100, or $5.00. The balance at the end of 6 months is $105.00. Over the second 6 months we earn 5% of $105.00, or $5.25. The balance at the end of one year is

$$\text{Balance after 1 year} = \$110.25.$$

We have earned a total of $10.25 in interest. As a percentage of $100, that is 10.25%. This number is the APY. It is the actual percent interest earned over the period of one year. The APY is always at least as large as the APR.

Here is a formula for the APY. In the formula, the APY and APR are in decimal form, and $n$ is the number of compounding periods per year.

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**Formula (4.4):** APY formula

$$\text{APY} = \left( 1 + \frac{\text{APR}}{n} \right)^n - 1.$$  

---

An algebraic derivation of this formula is shown in Algebraic Spotlight 4.1 below. Let’s apply this formula in the example above with 10% APR compounded semi-annually. Compounding is semi-annual so the number of periods is $n = 2$, and the APR as a decimal is 0.10:

$$\text{APY} = \left( 1 + \frac{0.10}{2} \right)^2 - 1$$

$$= \left( 1 + \frac{0.10}{2} \right)^2 - 1$$

$$= 0.1025.$$  

As a percent this is 10.25%—the same as we found above.
ALGEBRAIC SPOTLIGHT 4.1
Calculating APY

Suppose we invest money in an account that is compounded $n$ times per year. Then the period interest rate is

$$\text{Period rate} = \frac{\text{APR}}{n}.$$ 

The first step is to find the balance after 1 year. Using the compound interest formula (Formula 4.3), we find the balance to be

$$\text{Balance after 1 year} = \text{Principal} \times \left(1 + \frac{\text{APR}}{n}\right)^n.$$ 

How much money did we earn? We earned

$$\text{Balance minus Principal} = \text{Principal} \times \left(1 + \frac{\text{APR}}{n}\right)^n - \text{Principal}.$$ 

If we divide these earnings by the amount we started with, namely, the Principal, we get the percentage increase:

$$\text{APY as a decimal} = \left(1 + \frac{\text{APR}}{n}\right)^n - 1.$$ 

EXAMPLE 4.4  Calculating APY: an account with monthly compounding

We have an account that pays an APR of 10%. If interest is compounded monthly, find the APY. Round your answer as a percentage to two decimal places.

Solution:

We use the APY formula (Formula 4.4). As a decimal 10% is 0.10, and there are $n = 12$ compounding periods in a year. Therefore,

$$\text{APY} = \left(1 + \frac{\text{APR}}{n}\right)^n = \left(1 + \frac{0.10}{12}\right)^{12} - 1.$$ 

To four decimal places this is 0.1047. Thus the APY is about 10.47%. 

TRY IT YOURSELF 4.4

We have an account that pays an APR of 10%. If interest is compounded daily, find the APY. Round your answer as a percentage to two decimal places.

*The answer is provided at the end of this section.*

Using the APY

We can use the APY as an alternative to the APR for calculating compound interest. The following table gives both the APR and the APY for CDs from First Command Bank as of April 10, 2006.

*Table 4.1*
*Rates from First Command Bank*

<table>
<thead>
<tr>
<th>CD rates</th>
<th>APR</th>
<th>APY</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 Day</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1000 – $99,999.99</td>
<td>3.21%</td>
<td>3.25%</td>
</tr>
<tr>
<td>$100,000+</td>
<td>3.26%</td>
<td>3.30%</td>
</tr>
<tr>
<td>90 Day</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1000 – $99,999.99</td>
<td>3.22%</td>
<td>3.25%</td>
</tr>
<tr>
<td>$10,000 – $99,999.99</td>
<td>3.26%</td>
<td>3.30%</td>
</tr>
<tr>
<td>$100,000+</td>
<td>3.35%</td>
<td>3.40%</td>
</tr>
<tr>
<td>1 Year</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1000 – $99,999.99</td>
<td>3.31%</td>
<td>3.35%</td>
</tr>
<tr>
<td>$10,000 – $99,999.99</td>
<td>3.35%</td>
<td>3.40%</td>
</tr>
<tr>
<td>$100,000+</td>
<td>3.55%</td>
<td>3.60%</td>
</tr>
<tr>
<td>18 Month</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1000 – $99,999.99</td>
<td>3.56%</td>
<td>3.60%</td>
</tr>
<tr>
<td>$10,000 – $99,999.99</td>
<td>3.59%</td>
<td>3.65%</td>
</tr>
<tr>
<td>$100,000+</td>
<td>3.74%</td>
<td>3.80%</td>
</tr>
<tr>
<td>2 Year</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1000 – $99,999.99</td>
<td>3.80%</td>
<td>3.85%</td>
</tr>
<tr>
<td>$10,000 – $99,999.99</td>
<td>3.84%</td>
<td>3.90%</td>
</tr>
<tr>
<td>$100,000+</td>
<td>3.98%</td>
<td>4.05%</td>
</tr>
</tbody>
</table>

The APY can be used directly to calculate a year’s worth of interest. For example, suppose we purchase a one-year $100,000 CD from First Command Bank. Table 4.1 gives the APY for this CD as 3.60%. So after one year the interest we will earn is

\[
\text{One year’s interest} = 3.60\% \text{ of Principal} = 0.036 \times 100,000 = 3600.
\]
**EXAMPLE 4.5** Using APY to find value: CD

Suppose we purchase a one-year CD from First Command Bank for $25,000. What is the value of the CD at the end of the year?

**Solution:**

According to Table 4.1, the APY for this CD is 3.40%. This means that the CD earns 3.40% interest over the period of 1 year. Therefore,

\[
\text{One year’s interest} = 3.4\% \text{ of Principal} = 0.034 \times 25,000 = 850.
\]

We add this to the principal to find the balance:

\[
\text{Value after 1 year} = 25,000 + 850 = 25,850.
\]

**TRY IT YOURSELF 4.5**

Suppose we purchase a one-year CD from First Command Bank for $125,000. What is the value of the CD at the end of the year?

The answer is provided at the end of this section.

We can also use the APY rather than the APR to calculate the balance over several years if we wish. The APY tells us the actual percentage growth per year, including interest we earned during the year due to periodic compounding.\(^1\) Thus we can think of compounding annually (regardless of the actual compounding period) using the APY as the annual interest rate. Once again the balance is an exponential function. In this formula the APY is in decimal form.

**Formula (4.5): APY balance formula**

\[
\text{Balance after } t \text{ years} = \text{Principal} \times (1 + \text{APY})^t.
\]

\(^1\)The idea of an APY normally applies only to compound interest. It is not used for simple interest.
EXAMPLE 4.6  Using APY balance formula: CD balance at maturity

Suppose we earn 3.6% APY on a 10-year $100,000 CD. Calculate the balance at maturity.

Solution:
The APY in decimal form is 0.036 and the CD matures after 10 years, so we use $t = 10$ in the APY balance formula (Formula 4.5):

\[
\text{Balance after 10 years} = \text{Principal} \times (1 + \text{APY})^t
\]
\[
= 100,000 \times 1.036^{10}
\]
\[
= 142,428.71.
\]

TRY IT YOURSELF 4.6
Suppose we earn 4.1% APY on a 20-year $50,000 CD. Calculate the balance at maturity.

The answer is provided at the end of this section.

SUMMARY 4.2: APY

1. The APY gives the true (effective) annual interest rate. It takes into account money earned due to compounding.

2. If $n$ is the number of compounding periods per year,

\[
\text{APY} = \left(1 + \frac{\text{APR}}{n}\right)^n - 1.
\]

Here the APR and APY are both in decimal form.

3. The APY is always at least as large as the APR. When interest is compounded annually the APR and APY are equal. When compounding is more frequent, the APY is larger than the APR. The more frequent the compounding, the greater the difference.

4. The APY can be used to calculate the account balance after $t$ years:

\[
\text{Balance after } t \text{ years} = \text{Principal} \times (1 + \text{APY})^t.
\]

Here the APY is in decimal form.
Future and present value

Often we invest with a goal in mind, for example to make a down payment on the purchase of a car. The amount we invest is called the present value. The amount the account is worth after a certain period of time is called the future value of the original investment. Sometimes we know one of these two and would like to calculate the other.

**KEY CONCEPT**

The present value of an investment is the amount we initially invest. The future value is the value of that investment at some specified time in the future.

If the account grows only by compounding each period at a constant interest rate after we make an initial investment, then the present value is the principal, and the future value is the balance given by the compound interest formula (Formula 4.3):

\[
\text{Balance after } t \text{ periods} = \text{Principal} \times (1 + r)^t,
\]

so

\[
\text{Future value} = \text{Present value} \times (1 + r)^t.
\]

We can rearrange this formula to obtain

\[
\text{Present value} = \frac{\text{Future value}}{(1 + r)^t}.
\]

In these formulas \( t \) is the total number of compounding periods, and \( r \) is the period interest rate expressed as a decimal.

Which of these two formulas we should use depends on what question we are trying to answer. By the way, if we make regular deposits into the account then the formulas are much more complicated. We will examine that situation in Section 4.3.

*EXAMPLE 4.7 Calculating present and future value: investing for a car*

You would like to have $10,000 to buy a car three years from now. How much would you need to invest now in a savings account that pays an APR of 9% compounded monthly?
Solution:
In this problem you know the future value ($10,000) and would like to know the present value. The monthly rate is \( r = \frac{0.09}{12} = 0.0075 \) as a decimal, and the number of compounding periods is \( t = 36 \) months. Thus,

\[
\text{Present value} = \frac{\text{Future value}}{(1 + r)^t} = \frac{10,000}{1.0075^{36}} = 7641.49.
\]

Therefore, we should invest $7641.49 now.

TRY IT YOURSELF 4.7
Find the future value of an account after 4 years if the present value is $900, the APR is 8%, and interest is compounded quarterly.

The answer is provided at the end of this section.

Doubling time for investments
Exponential functions eventually get very large. This means that even a modest investment today in an account that pays compound interest will grow to be very large in the future. In fact your money will eventually double and then double again.

In Section 3 of Chapter 3 we used logarithms to find the doubling time. Now we introduce the Rule of 72 as a quick way of estimating the doubling time.
SUMMARY 4.3: DOUBLING TIME REVISITED

- The exact doubling time is given by the formula

\[
\text{Number of periods to double} = \frac{\log 2}{\log(1 + r)} = \frac{\log 2}{\log(1 + r)}.
\]

Here \( r \) is the period interest rate as a decimal.

- We can approximate the doubling time using the Rule of 72:

\[
\text{Estimate for doubling time} = \frac{72}{\text{APR}}.
\]

Here the APR is expressed as a percentage, not as a decimal, and time is measured in years. The estimate is fairly accurate if the APR is 15% or less, but we emphasize that this is only an approximation.

EXAMPLE 4.8 Computing doubling time: an account with quarterly compounding

Suppose an account earns an APR of 8% compounded quarterly. First estimate the doubling time using the Rule of 72. Then calculate the exact doubling time and compare the result with your estimate.

Solution:

The Rule of 72 gives the estimate

\[
\text{Estimate for doubling time} = \frac{72}{\text{APR}} = \frac{72}{8} = 9 \text{ years}.
\]

To find the exact doubling time, we need the period interest rate \( r \). The period is a quarter, so \( r = \frac{0.08}{4} = 0.02 \). Putting this result into the doubling time formula, we find

\[
\text{Number of periods to double} = \frac{\log 2}{\log(1 + r)}
\]

\[
= \frac{\log 2}{\log(1 + 0.02)}.
\]

The result is about 35.0. Therefore, the actual doubling time is 35.0 quarters, or 8 years and 9 months. Our estimate of 9 years was 3 months too high.

TRY IT YOURSELF 4.8

Suppose an account has an APR of 12% compounded monthly. First estimate the
doubling time using the Rule of 72. Then calculate the exact doubling time in years and months.

*The answer is provided at the end of this section.*

---

**Try It Yourself Answers**

**TRY IT YOURSELF 4.1** Calculating simple interest: an account: $540

**TRY IT YOURSELF 4.2** Calculating compound interest: annual compounding: $631.24

**TRY IT YOURSELF 4.4** Calculating APY: an account with monthly compounding: About 10.52%

**TRY IT YOURSELF 4.5** Using APY to find value: CD: $129,500

**TRY IT YOURSELF 4.6** Using APY balance formula: CD balance at maturity: $111,682.36

**TRY IT YOURSELF 4.7** Calculating present and future value: investing for a car: $1235.51

**TRY IT YOURSELF 4.8** Computing doubling time: an account with quarterly compounding: The Rule of 72 gives an estimate of 6 years. The exact formula gives 69.7 months, or about 5 years and 10 months.

---

**Exercise Set 4.1**

In exercises for which you are asked to calculate the APR or APY as the final answer, round your answer as a percentage to two decimal places.

1. **Simple interest**: Assume a 3-month CD purchased for $2000 pays simple interest at an annual rate of 10%. How much total interest does it earn? What is the balance at maturity?

2. **More simple interest**: Assume a 30-month CD purchased for $3000 pays simple interest at an annual rate of 5.5%. How much total interest does it earn? What is the balance at maturity?
3. **Make a table:** Suppose you put $3000 in a savings account at an APR of 8% compounded quarterly. Fill in the table below. (Calculate the interest and compound it by hand each quarter rather than using the compound interest formula.)

<table>
<thead>
<tr>
<th>Quarter</th>
<th>Interest earned</th>
<th>Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$</td>
<td>$</td>
</tr>
<tr>
<td>2</td>
<td>$</td>
<td>$</td>
</tr>
<tr>
<td>3</td>
<td>$</td>
<td>$</td>
</tr>
<tr>
<td>4</td>
<td>$</td>
<td>$</td>
</tr>
</tbody>
</table>

4. **Make another table:** Suppose you put $4000 in a savings account at an APR of 6% compounded monthly. Fill in the table below. (Calculate the interest and compound it by hand each month rather than using the compound interest formula.)

<table>
<thead>
<tr>
<th>Month</th>
<th>Interest earned</th>
<th>Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$</td>
<td>$</td>
</tr>
<tr>
<td>2</td>
<td>$</td>
<td>$</td>
</tr>
<tr>
<td>3</td>
<td>$</td>
<td>$</td>
</tr>
</tbody>
</table>

**Compound interest calculated by hand:** Assume we invest $2000 for one year in a savings account that pays an APR of 10% compounded quarterly. Exercises 5 through 7 refer to this account.

5. Make a table to show how much is in the account at the end of each quarter. (Calculate the interest and compound it by hand each quarter rather than using the compound interest formula.)

6. Use your answer to Exercise 5 to determine how much total interest the account has earned after 1 year.

7. Compare the earnings to what simple interest or semi-annual compounding would yield.

8. **Using the compound interest formula:** This is a continuation of Exercise 5. In Exercise 5 we invested $2000 for one year in a savings account with an APR of 10% compounded quarterly. Apply the compound interest formula (Formula 4.3) to see whether it gives the answer for the final balance obtained in Exercise 5.
9. **The difference between simple and compound interest:** Suppose you invest $1000 in a savings account that pays an APR of 6%. If the account pays simple interest, what is the balance in the account after 20 years? If interest is compounded monthly, what is the balance in the account after 20 years?

**Calculating interest:** Assume an investment of $7000 earns an APR of 6% compounded monthly for 18 months. Exercises 10 and 11 refer to this investment.

10. How much money is in your account after 18 months?

11. How much interest has been earned?

12. **Getting rich:** Assume an investment of $100 earns an APR of 5% compounded annually. Calculate the balance after 190 years. Would you feel cheated if you had paid to attend the seminar depicted in the accompanying cartoon?

13. **Retirement options:** At age 25 you start work for a company and are offered two retirement options.

   **Retirement option 1:** When you retire, you will receive a lump sum of $30,000 for each year of service.

   **Retirement option 2:** When you start to work, the company deposits $15,000 into an account that pays a monthly interest rate of 1%, and interest is compounded monthly. When you retire you get the balance of the account.

Which option is better if you retire at age 65? Which is better if you retire at age 55?

14. **Compound interest:** Assume an 18-month CD purchased for $7000 pays an APR of 6% compounded monthly. What is the APY? Would the APY change if the investment were $11,000 for 30 months with the same APR and with monthly compounding?

15. **More compound interest:** Assume a 24-month CD purchased for $7000 pays an APY of 4.25% (and of course interest is compounded). How much do you have at maturity?
16. **Interest and APY:** Assume a 1-year CD purchased for $2000 pays an APR of 8% that is compounded semi-annually. How much is in the account at the end of each compounding period? (Calculate the interest and compound it by hand each period rather than using the compound interest formula.) How much total interest does it earn? What's the APY?

17. **More interest and APY:** Assume a 1-year CD purchased for $2000 pays an APR of 8% that is compounded quarterly. How much is in the account at the end of each compounding period? (Calculate the interest and compound it by hand each period rather than using the compound interest formula.) How much total interest does it earn? What's the APY?

**A CD from First Command Bank:** Suppose you buy a 2-year CD for $10,000 from First Command Bank. Exercises 18 through 20 refer to this CD.

18. Use the APY from Table 4.1 to determine how much interest it earns for you at maturity.

19. Assume monthly compounding and use the APY formula in part 2 of Summary 4.2 to find the APY from the APR. Compare this to the APY in the table.

20. Assume monthly compounding. Use the APR in the table and the compound interest formula to determine how much interest the CD earns for you at maturity.

21. **Some interest and APY calculations:** Parts b and c refer to the rates at First Command Bank shown in Table 4.1.

   (a) Assume that a 1-year CD for $5000 pays an APR of 8% that is compounded quarterly. How much total interest does it earn? What is the APY?

   (b) If you purchase a 1-year CD for $150,000 from First Command Bank how much interest will you receive at maturity? Is compounding taking place? Explain.

   (c) If you purchase a 2-year CD for $150,000 from First Command Bank the APY (4.05%) is greater than the APR (3.98%) because compounding is taking place. We are not told, however, what the compounding period is. Use the APR to calculate what the APY would be with monthly compounding. How does your answer compare to the APY in the table?
True or false: In Exercises 22 through 27, answer “True” if the statement is always true and “False” otherwise. If it is false, explain why.

22. Principal is the amount you have in your account.

23. If the APY is greater than the APR, the cause must be compounding.

24. The APY can be less than the APR over a short period of time.

25. If the interest earned by a savings account is compounded semi-annually, equal amounts of money are deposited twice during the year.

26. If the interest earned by a savings account paying an APR of 8% is compounded quarterly, 2% of the current balance is deposited four times during the year.

27. If a savings account pays an APR of 8% compounded monthly, 8% of the balance is deposited each month during the year.

28. Interest and APR: Assume that a 2-year CD for $4000 pays an APY of 8%. How much interest will it earn? Can you determine the APR?

29. Find the APR: Sue bought a 6-month CD for $3000. She said that at maturity it paid $112.50 in interest. Assume this was simple interest, and determine the APR.

First Command Bank: Exercises 30 through 32 refer to the rates at First Command Bank shown in Table 4.1.

30. Larry invests $99,999 and Sue invests $100,000, each for one year. How much more did Sue earn than Larry?

31. Compute the amount earned on $100,000 invested at 3.26% APR in one year if compounding is taking place daily.

32. Considering the results of Exercise 31 and the APY given in the table, how often do you think First Command Bank is compounding interest on 30-day $100,000 CDs?

33. Future value: What is the future value of a 10-year investment of $1000 at an APR of 9% compounded monthly? Explain what your answer means.
34. **Present value:** What is the present value of an investment that will be worth $2000 at the end of 5 years? Assume an APR of 6% compounded monthly. Explain what your answer means.

35. **Doubling again:** You have invested $2500 at an APR of 9%. Use the Rule of 72 to estimate how long it will be until your investment reaches $5000, and how long it will be until your investment reaches $10,000.

36. **Getting APR from doubling rate:** A friend tells you that her savings account doubled in 12 years. Use the Rule of 72 to estimate what the APR of her account was.

37. **Find the doubling time:** Consider an investment of $3000 at an APR of 6% compounded monthly. Use the formula that gives the exact doubling time to determine exactly how long it will take for the investment to double. (See the first part of Summary 4.3. Be sure to use the monthly rate for \( r \).) Express your answer in years and months. Compare this result with the estimate obtained from the Rule of 72.

The following exercises are designed to be solved using technology such as calculators or computer spreadsheets. For assistance see the technology supplement.

38. **Find the rate:** According to an article by Doug Abrahms in the *Reno Gazette-Journal* on June 22, 2004, the U.S. House of Representatives passed a bill to distribute funds to members of the Western Shoshone tribe. The article says:

> The Indian Claims Commission decided the Western Shoshone lost much of their land to gradual encroachment. The tribe was awarded $26 million in 1977. That has grown to about $145 million through compound interest, but the tribe never took the money.

Assume monthly compounding and determine the APR that would give this growth in the award over the 27 years from 1977 to 2004. *Note:* This can also be solved without technology using algebra.
39. **Solve for the APR:** Suppose a CD advertises an APY of 8.5%. Assuming the APY was a result of monthly compounding, solve the equation

\[
0.085 = \left(1 + \frac{\text{APR}}{12}\right)^{12} - 1
\]

to find the APR. *Note:* This can also be solved without technology using algebra.

40. **Find the compounding period:** Suppose a CD advertises an APR of 5.10% and an APY of 5.20%. Solve the equation

\[
0.052 = \left(1 + \frac{0.051}{n}\right)^n - 1
\]

for \(n\) to determine how frequently interest is compounded.
4.2 BORROWING: HOW MUCH CAR CAN YOU AFFORD?

The following article appeared in U.S. News and World Report.

NEWSPAPER ARTICLE 4.2: PAYING FOR COLLEGE ON THE INSTALLMENT PLAN
SHAHEENA AHMAD
SEPTEMBER 7, 1998

Because the “amount due” at the start of each semester can be so daunting, hundreds of colleges now offer families the option of paying on an installment plan. The payment programs typically let families cover tuition, room, and board—or whatever is left of the bill after financial aid awards have been factored in—in equal monthly installments over the course of 10 or 12 months.

The delayed payments are generally interest free, although participants often pay an nominal fee each year (usually $40 to $75) to enroll. A few schools impose a finance charge instead; the University of Miami, for example, charges a fee of 3 percent of the amount to be paid in installments, or $300 a year for a family that finances $10,000. At most schools, any student, regardless of financial need, can participate in the delayed payment program. Some institutions, such as Georgetown and Johns Hopkins, administer the programs themselves; parents send their monthly checks directly to the bursar’s office.

Most colleges, however, contract with third parties to handle the payments. Schools that have arrangements with these organizations—which include Academic Management Services, Key education Resources, and Tuition Management Systems—often send out enrollment information about the plans when they notify students of financial aid awards. Parents who sign up receive a coupon book or a billing statement each month and send their checks to the outside service-starting almost immediately after their son or daughter graduates from high school.

Installment payments are a part of virtually everyone’s life. The goal of this section is to explain how such payments are calculated and to see the implications of installment plans.

Installment loans and the monthly payment

When we borrow money to buy a car or a house, the lending institution typically requires that we pay back the loan plus interest by making the same payment each month for a certain number of months.
KEY CONCEPT

With an **installment loan** you borrow money for a fixed period of time, called the **term** of the loan, and you make regular payments (usually monthly) to pay off the loan plus interest in that time.

To see how the monthly payment is calculated we consider a simple example. Suppose you need $100 to buy a calculator but don’t have the cash available. Your sister is willing, however, to lend you the money at a rate of 5% per month provided you repay it with equal payments over the next two months. (By the way, this is an astronomical APR of 60%.)

How much must you repay each month? Because the loan is for only two months, you need to repay at least half of the $100, or $50, each of the two months. But that doesn’t account for the interest. If we were to pay off the loan in a lump sum at the end of the two-month term, the interest on the account would be calculated in the same way as for a savings account that pays 5% per month. Using the compound interest formula (Formula 4.3 from the preceding section), we find

\[
\text{Balance owed after 2 months} = \text{Principal} \times (1 + r)^t = 100 \times 1.05^2 = 110.25
\]

This would be two monthly payments of approximately $55.13. This amount overestimates your monthly payment, however: In the second payment you shouldn’t have to pay interest on the amount you have already repaid.

We will see shortly that the correct payment is $53.78 for each of the two months. This is not an obvious amount, but it is reasonable—it lies between the two extremes of $50 and $55.13.

When we take out an installment loan, the amount of the payment depends on three things: the amount of money we borrow (sometimes called the principal), the interest rate (or APR), and the term of the loan.

Each payment reduces the balance owed, but at the same time interest is accruing on the outstanding balance. This makes the calculation of the monthly payment fairly complicated, as you might surmise from the following formula.\(^2\)

\(^2\)This formula is built in as a feature on some hand-held calculators and most computer spreadsheets. Loan calculators are also available on the Web.
**Formula (4.6): Monthly payment formula**

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{((1 + r)^t - 1)}
\]

Here \( t \) is the term in months and \( r = \text{APR}/12 \) is the monthly interest rate as a decimal.

A derivation of this formula is presented at the end of this section in Algebraic Spotlight 4.2 and Algebraic Spotlight 4.3.

If you want to borrow money, the monthly payment formula allows you to determine in advance whether you can afford that car or home you want to buy. The formula also lets you check the accuracy of any figure that a potential lender quotes.

Let’s return to the earlier example of a $100 loan to buy a calculator. We have a monthly rate of 5%. Expressed as a decimal, 5% is 0.05, so \( r = 0.05 \) and \( 1 + r = 1.05 \). Because we pay off the loan in 2 months, we use \( t = 2 \) in the monthly payment formula (Formula 4.6):

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{((1 + r)^t - 1)}
\]

\[
= \frac{100 \times 0.05 \times 1.05^2}{(1.05^2 - 1)} = \frac{53.78}{1.01005}
\]

\[= \$53.78.\]

**EXAMPLE 4.9 Using the monthly payment formula: college loan**

You need to borrow $5000 so you can attend college next fall. You get the loan at an APR of 6% to be paid off in monthly installments over 3 years. Calculate your monthly payment.

**Solution:**

The monthly rate as a decimal is

\[r = \text{Monthly rate} = \frac{\text{APR}}{12} = \frac{0.06}{12} = 0.005.\]

This gives \( 1 + r = 1.005 \). We want to pay off the loan in 3 years or 36 months, so we
use a term of $t = 36$ in the monthly payment formula:

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{(1 + r)^t - 1}
\]

\[
= \frac{5000 \times 0.005 \times 1.005^{36}}{(1.005^{36} - 1)}
\]

\[
= 152.11.
\]

---

**TRY IT YOURSELF 4.9**

You borrow $8000 at an APR of 9% to be paid off in monthly installments over 4 years. Calculate your monthly payment.

*The answer is provided at the end of this section.*

---

Suppose you can afford a certain monthly payment and you’d like to know how much you can borrow to stay within that budget. The monthly payment formula can be rearranged to answer that question.

**Formula (4.7): Companion monthly payment formula**

\[
\text{Amount borrowed} = \frac{\text{Monthly payment} \times ((1 + r)^t - 1)}{(r \times (1 + r)^t)}.
\]

---

**EXAMPLE 4.10 Computing how much I can borrow: buying a car**

We can afford to make payments of $250 per month for 3 years. Our car dealer is offering us a loan at an APR of 5%. What price automobile should we be shopping for?

**Solution:**

The monthly rate as a decimal is

\[
r = \text{Monthly rate} = \frac{0.05}{12}.
\]

To four decimal places this is 0.0042, but for better accuracy we won’t round this number. Now 3 years is 36 months, so we use $t = 36$ in the companion payment formula.
formula (Formula 4.7):

\[
\text{Amount borrowed} = \frac{\text{Monthly payment} \times ((1 + r)^t - 1)}{(r \times (1 + r)^t)}
\]

\[
= \frac{250 \times ((1 + 0.05/12)^{36} - 1)}{(0.05/12) \times (1 + 0.05/12)^{36}}
\]

\[= \$8341.43.\]

We should shop for cars that cost $8341.43 or less.

---

**TRY IT YOURSELF 4.10**

We can afford to make payments of $300 per month for 4 years. We can get a loan at an APR of 4%. How much money can we afford to borrow?

*The answer is provided at the end of this section.*

---

**SUMMARY 4.4: MONTHLY PAYMENTS**

In parts 1 and 2, the monthly rate \( r \) is the APR in decimal form divided by 12, and \( t \) is the term in months.

1. The monthly payment is

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{((1 + r)^t - 1)}.
\]

2. A companion formula gives the amount borrowed in terms of the monthly payment:

\[
\text{Amount borrowed} = \frac{\text{Monthly payment} \times ((1 + r)^t - 1)}{(r \times (1 + r)^t)}.
\]

3. These formulas are sensitive to round-off error, so it is best do calculations all at once, keeping all the decimal places rather than doing parts of a computation and entering the rounded numbers.
EXAMPLE 4.11 Calculating monthly payment and amount borrowed: a new car

Suppose we need to borrow $15,000 at an APR of 9% to buy a new car.

a. What will the monthly payment be if we borrow the money for $\frac{3}{2}$ years? How much interest will we have paid by the end of the loan?

b. We find that we cannot afford the $15,000 car because we can only afford a monthly payment of $300. What price car can we shop for if the dealer offers a loan at a 9% APR for a term of $\frac{3}{2}$ years?

Solution:

a. The monthly rate as a decimal is

$$r = \text{Monthly rate} = \frac{\text{APR}}{12} = \frac{0.09}{12} = 0.0075.$$  

We are paying off the loan in $\frac{3}{2}$ years, so $t = 3.5 \times 12 = 42$ months. Therefore,

$$\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{((1 + r)^t - 1)} = \frac{15,000 \times 0.0075 \times 1.0075^{42}}{(1.0075^{42} - 1)} = 417.67.$$  

Now let’s find the amount of interest paid. We will make 42 payments of $417.67 for a total of $42 \times 417.67 = 17,542.14$. Because the amount we borrowed is $15,000$, that means that the total amount of interest paid is $17,542.14 - 15,000 = 2,542.14$.

b. The monthly interest rate as a decimal is $r = 0.0075$, and there are still 42 payments. We know that the monthly payment we can afford is $300$. We use the companion formula (Formula 4.7) to find the amount we can borrow on this budget:

$$\text{Amount borrowed} = \frac{\text{Monthly payment} \times ((1 + r)^t - 1)}{(r \times (1 + r)^t)} = \frac{300 \times (1.0075^{42} - 1)}{(0.0075 \times 1.0075^{42})} = 10,774.11.$$  

This means that we can afford to shop for a car that costs no more than $10,774.11$. 
The next example shows how saving compares with borrowing.

**EXAMPLE 4.12** Comparing saving versus borrowing: a loan and a CD

**a.** Suppose we borrow $5000 for one year at an APR of 7.5%. What will the monthly payment be? How much interest will we have paid by the end of the year?

**b.** Suppose we buy a 1-year $5000 CD at an APR of 7.5% compounded monthly. How much interest will we be paid at the end of the year?

**c.** In part a the financial institution loaned us $5000 for one year, but in part b we loaned the financial institution $5000 for one year. What is the difference in the amount of interest paid? Explain why the amounts are different.

**Solution:**

**a.** In this case the principal is $5000, the monthly interest rate $r$ as a decimal is $0.075/12 = 0.00625$, and the number $t$ of payments is 12. We use the monthly payment formula (Formula 4.6):

$$\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1+r)^t}{(1+r)^t - 1}$$

$$= \frac{5000 \times 0.00625 \times 1.00625^{12}}{(1.00625^{12} - 1)}$$

$$= \$433.79.$$

We will make 12 payments of $433.79 for a total of $12 \times 433.79 = 5205.48. Because the amount we borrowed is $5000, the total amount of interest paid is $205.48.

**b.** To calculate the interest earned on the CD, we use the compound interest formula (Formula 4.3 from the preceding section). The monthly rate is the same as in part a. Therefore,

$$\text{Balance} = \text{Principal} \times (1+r)^t$$

$$= 5000 \times 1.00625^{12}$$

$$= \$5388.16.$$

This means that the total amount of interest we earned is $388.16.
c. The interest earned on the $5000 CD is $182.68 more than the interest paid on the $5000 loan in part a.

Here is the explanation for this difference: When we save money, the financial institution credits our account with an interest payment—in this case every month. We continue to earn interest on the full $5000 and on those interest payments for the entire year. When we borrow money, however, we repay the loan monthly, thus decreasing the balance owed. We are being charged interest only on the balance owed, not on the full $5000 we borrowed.

One common piece of financial advice is to reduce the amount you borrow by making a down payment. Exercises 15 through 17 explore the effect of making a down payment on the monthly payment and the total interest paid.

**Estimating payments for short-term loans**

The formula for the monthly payment is complicated, and it’s easy to make a mistake in the calculation. Is there a simple way to give an estimate for the monthly payment? There are in fact a couple of ways to do this. We give one of them here and another when we look at home mortgages.

One obvious estimate for a monthly payment is to divide the loan amount by the term (in months) of the loan. This would be our monthly payment if no interest were charged. For a loan with a relatively short term and not too large an interest rate, this gives a rough lower estimate for the monthly payment.³

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**RULE OF THUMB 4.2: MONTHLY PAYMENTS FOR SHORT-TERM LOANS**

For all loans the monthly payment is at least the amount we would pay each month if no interest were charged, which is the amount of the loan divided by the term (in months) of the loan. This would be the payment if the APR were 0%. It’s a rough estimate of the monthly payment for a short-term loan if the APR is not large.

³If the term is at most 5 years and the APR is less than 7.5%, the actual payment is within 20% of the ratio. If the term is at most 2 years and the APR is less than 9%, the actual payment is within 10% of the ratio.
CHAPTER 4 Personal Finance

**EXAMPLE 4.13** Estimating monthly payment: can we afford it?

The largest monthly payment we can afford is $800. Can we afford to borrow a principal of $20,000 with a term of 24 months?

**Solution:**
The rule of thumb says that the monthly payment is at least $20,000/24 = $833.33. This is more than $800, so we can’t afford this loan.

**TRY IT YOURSELF 4.13**
The largest monthly payment we can afford is $450. Can we afford to borrow a principal of $18,000 with a term of 36 months?

*The answer is provided at the end of this section.*

For the loan in the preceding example our rule of thumb says that the monthly payment is at least $833.33. Remember that this amount does not include the interest payments. If the loan has an APR of 6.6%, for example, the actual payment is $891.83.

Once again, a rule of thumb gives an estimate—not the exact answer. It can at least tell us quickly whether we should be shopping on the BMW car lot.

**Amortization tables and equity**

When you make payments on an installment loan, part of each payment goes toward interest, and part goes toward reducing the balance owed. An *amortization table* is a running tally of payments made and the outstanding balance owed.

**KEY CONCEPT**

An amortization table or amortization schedule for an installment loan shows for each payment made the amount applied to interest, the amount applied to the balance owed, and the outstanding balance.

**EXAMPLE 4.14** Making an amortization table: buying a computer

Suppose we borrow $1000 at 12% APR to buy a computer. We pay off the loan in 12 monthly payments. Make an amortization table showing payments over the
first 6 months.

**Solution:**

The monthly rate is \(12\%/12 = 1\%\). As a decimal this is \(r = 0.01\). The monthly payment formula with \(t = 12\) gives

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{(1 + r)^t - 1} = \frac{1000 \times 0.01 \times 1.01^{12}}{(1.01^{12} - 1)} = 88.85.
\]

Because the monthly rate is 1%, each month we pay 1% of the outstanding balance in interest. When we make our first payment the outstanding balance is $1000, so we pay 1% of $1000, or $10.00, in interest. Thus $10 of our $88.85 goes toward interest, and the remainder, $78.85, goes toward the outstanding balance. So after the first payment we owe:

\[
\text{Balance owed after 1 payment} = 1000.00 - 78.85 = 921.15.
\]

When we make a second payment, the outstanding balance is $921.15. We pay 1% of $921.15 or $9.21 in interest, so $88.85 - 9.21 = 79.64 goes toward the balance due. This gives the balance owed after the second payment:

\[
\text{Balance owed after 2 payments} = 921.15 - 79.64 = 841.51.
\]

If we continue in this way we get the following table.

<table>
<thead>
<tr>
<th>Payment number</th>
<th>Payment</th>
<th>Applied to interest</th>
<th>Applied to balance owed</th>
<th>Outstanding balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$88.85</td>
<td>1% of $1000.00 = $10.00</td>
<td>$78.85</td>
<td>$1000.00</td>
</tr>
<tr>
<td>2</td>
<td>$88.85</td>
<td>1% of $921.15 = $9.21</td>
<td>$79.64</td>
<td>$841.51</td>
</tr>
<tr>
<td>3</td>
<td>$88.85</td>
<td>1% of $841.51 = $8.42</td>
<td>$80.43</td>
<td>$761.08</td>
</tr>
<tr>
<td>4</td>
<td>$88.85</td>
<td>1% of $761.08 = $7.61</td>
<td>$81.24</td>
<td>$679.84</td>
</tr>
<tr>
<td>5</td>
<td>$88.85</td>
<td>1% of $679.84 = $6.80</td>
<td>$82.05</td>
<td>$597.79</td>
</tr>
<tr>
<td>6</td>
<td>$88.85</td>
<td>1% of $597.79 = $5.98</td>
<td>$82.87</td>
<td>$514.92</td>
</tr>
</tbody>
</table>

**TRY IT YOURSELF 4.14**

Suppose we borrow $1000 at 30% APR and pay it off in 24 monthly payments. Make an amortization table showing payments over the first 3 months.
The answer is provided at the end of this section.

The loan in Example 4.14 is used to buy a computer. The amount you have paid toward the actual cost of the computer (the principal) at a given time is referred to as your equity in the computer. For example, the table above tells us that after 4 payments we still owe $679.84. That means we have paid a total of $1000.00 − $679.84 = $320.16 toward the principal. That is our equity in the computer.

KEY CONCEPT

If you borrow money to pay for an item, your equity in that item at a given time is the part of the principal you have paid.

EXAMPLE 4.15 Calculating equity: buying land

You borrow $150,000 at an APR of 6% to purchase a plot of land. You pay off the loan in monthly payments over 10 years.

a. Find the monthly payment.

b. Complete the 4-month amortization table below.

<table>
<thead>
<tr>
<th>Payment number</th>
<th>Payment</th>
<th>Applied to interest</th>
<th>Applied to balance owed</th>
<th>Outstanding balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>$150,000.00</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. What is your equity in the land after 4 payments?

Solution:

a. The monthly rate is APR/12 = 6%/12 = 0.5%. As a decimal this is \( r = 0.005 \). We use the monthly payment formula with \( t = 10 \times 12 = 120 \) months:

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{(1 + r)^t - 1} = \frac{150,000 \times 0.005 \times 1.005^{120}}{(1.005^{120} - 1)} = 1665.31.
\]
b. For the first month we pay 0.5% of the outstanding balance of $150,000:

First month interest = $150,000 \times 0.005 = $750.

Now $750 of the $1665.31 payment goes to interest and the remainder, $1665.31 – $750 = $915.31, goes toward reducing the principal. The balance owed after one month is

Balance owed after 1 month = $150,000 – $915.31 = $149,084.69.

This gives the first row of the table. The completed table is shown below.

<table>
<thead>
<tr>
<th>Payment number</th>
<th>Payment</th>
<th>Applied to interest</th>
<th>Applied to balance owed</th>
<th>Outstanding balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1665.31</td>
<td>0.5% of $150,000.00 = $750.00</td>
<td>$915.31</td>
<td>$149,084.69</td>
</tr>
<tr>
<td>2</td>
<td>$1665.31</td>
<td>0.5% of $149,084.69 = $745.42</td>
<td>$919.89</td>
<td>$148,164.80</td>
</tr>
<tr>
<td>3</td>
<td>$1665.31</td>
<td>0.5% of $148,164.80 = $740.82</td>
<td>$924.49</td>
<td>$147,240.31</td>
</tr>
<tr>
<td>4</td>
<td>$1665.31</td>
<td>0.5% of $147,240.31 = $736.20</td>
<td>$929.11</td>
<td>$146,311.20</td>
</tr>
</tbody>
</table>

c. The table from part b tells us that after 4 payments we still owe $146,311.21. So our equity is

Equity after 4 months = $150,000 – $146,311.21 = $3688.79.

The graph in Figure 4.4 shows the percentage of each payment from Example 4.15 that goes toward interest, and Figure 4.5 shows how equity is built. Note that in the early months a large percentage of the payment goes toward interest. For long-term loans an even larger percentage of the payment goes toward interest early on. This means that equity is built slowly at first. The rate of growth of equity increases over the life of the loan. Note in Figure 4.5 that when you have made half of the payments (60 payments) you have built an equity of just over $60,000—much less than half of the purchase price.
Home mortgages

A home mortgage is a loan for the purchase of a home. It is very common for a mortgage to last as long as 30 years. The early mortgage payments go almost entirely toward interest, with a small part going to reduce the principal. As a consequence your home equity grows very slowly. For a 30-year mortgage of $150,000 at an APR of 6%, Figure 4.6 shows the percentage of each payment that goes to interest, and Figure 4.7 shows the equity built. Your home equity is very important because it tells you how much money you can actually keep if you sell your house, and it can also be used as collateral to borrow money.

**EXAMPLE 4.16 Computing interest: 30-year mortgage**

Your neighbor took out a 30-year mortgage for $300,000 at an APR of 9%. She says that she will wind up paying more in interest than for the home (that is, the principal). Is that true?
Solution:
We first need to find the monthly payment. The monthly rate as a decimal is

\[ r = \text{Monthly rate} = \frac{\text{APR}}{12} = \frac{0.09}{12} = 0.0075. \]

Because the loan is for 30 years, we use \( t = 30 \times 12 = 360 \) months in the monthly payment formula:

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{(1 + r)^t - 1} = $300,000 \times 0.0075 \times 1.0075^{360} \]
\[
= \frac{(1.0075^{360} - 1)}{(1.0075^{360} - 1)} = $2413.87.
\]

She will make 360 payments of $2413.87, for a total of

\[ \text{Total amount paid} = 360 \times $2413.87 = $868,993.20. \]

The interest paid is the excess over $300,000, or $568,993.20. Your neighbor paid almost twice as much in interest as she did for the home.

**TRY IT YOURSELF 4.16**
Find the interest paid on a 25-year mortgage of $450,000 at an APR of 7.2%.
*The answer is provided at the end of this section.*

Now we see the effect on the monthly payment of varying the term of the mortgage.

**EXAMPLE 4.17  Determining monthly payment and term: choices for mortgages**

You need to secure a loan of $250,000 to purchase a home. Your lending institution offers you three options:

**Option 1**: A 30-year mortgage at 8.4% APR.

**Option 2**: A 20-year mortgage at 7.2% APR.

**Option 3**: A 30-year mortgage at 7.2% APR, including a fee of 4 loan points.
*Note: “Points” are a fee you pay for the loan in return for a decrease in the interest rate. In this case a fee of 4 points means you pay 4% of the loan, or
$10,000. One way to do this is to borrow the fee from the bank by just adding the $10,000 to amount you borrow. The bank keeps the $10,000 and the other $250,000 goes to buy the home.

Determine the monthly payment for each of these options.

Solution:

Option 1: The monthly rate as a decimal is

\[
r = \text{Monthly rate} = \frac{\text{APR}}{12} = \frac{0.084}{12} = 0.007.
\]

We use the monthly payment formula with \( t = 360 \) months:

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{((1 + r)^t - 1)} = \frac{250,000 \times 0.007 \times 1.007^{360}}{(1.007^{360} - 1)} = \$1,904.59.
\]

Option 2: The APR for a 20-year loan is lower. (It is common for loans with shorter terms to have lower interest rates.) An APR of 7.2% is a monthly rate of \( r = 0.072/12 = 0.006 \) as a decimal. We use the monthly payment formula with \( t = 20 \times 12 = 240 \) months:

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{((1 + r)^t - 1)} = \frac{250,000 \times 0.006 \times 1.006^{240}}{(1.006^{240} - 1)} = \$1,968.37.
\]

The monthly payment is about $60 higher than for the 30-year mortgage, but you pay the loan off in 20 years rather than 30 years.

Option 3: Adding 4% to the amount we borrow gives a new loan amount of $260,000. As with Option 2, the APR of 7.2% gives a monthly rate of \( r = 0.006 \). The term of 30 years means that we put \( t = 360 \) months into the monthly payment formula:

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{((1 + r)^t - 1)} = \frac{260,000 \times 0.006 \times 1.006^{360}}{(1.006^{360} - 1)} = \$1,764.85.
\]
Getting the lower interest rate makes a big difference in the monthly payment, even with the extra 4% added to the original loan balance. This is clearly a better choice than Option 1 if we consider only the monthly payment. Option 3 does require borrowing more money than Option 1 and could have negative consequences if you need to sell the home early. Further, comparing the amount of interest paid shows that Option 2 is the best choice from that point of view—if you can afford the monthly payment.

Adjustable-rate mortgages

In the summer of 2007 a credit crisis involving home mortgages had dramatic effects on many people and ultimately on the global economy. A significant factor in the crisis was the widespread use of adjustable-rate mortgages or ARMs.

**KEY CONCEPT**

A fixed-rate mortgage keeps the same interest rate over the life of the loan. In the case of an adjustable-rate mortgage or ARM, the interest rate may vary over the life of the loan. The rate is often tied to the prime interest rate, which is the rate banks must pay to borrow money.

One advantage of an ARM is that the initial rate is often lower than the rate for a comparable fixed-rate mortgage. One disadvantage of an ARM is that a rising prime interest rate may cause significant increases in the monthly payment.

**EXAMPLE 4.18**  Comparing monthly payments: fixed-rate and adjustable-rate mortgages

We want to borrow $200,000 for a 30-year home mortgage. We have found an APR of 6.6% for a fixed-rate mortgage and an APR of 6% for an ARM. Compare the initial monthly payments for these loans.

**Solution:**

Both loans have a principal of $200,000 and a term of $t = 360$ months. For the
fixed-rate mortgage, the monthly rate in decimal form is \( r = \frac{0.066}{12} = 0.0055 \).
The monthly payment formula gives

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{(1 + r)^t - 1}
\]

\[
= \frac{200,000 \times 0.0055 \times 1.0055^{360}}{(1.0055^{360} - 1)}
\]

\[
= \$1277.32.
\]

For the ARM, the initial monthly rate in decimal form is \( r = \frac{0.06}{12} = 0.005 \).
The monthly payment formula gives

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{(1 + r)^t - 1}
\]

\[
= \frac{200,000 \times 0.005 \times 1.005^{360}}{(1.005^{360} - 1)}
\]

\[
= \$1199.10.
\]

The initial monthly payment for the ARM is almost \$80 less than the payment for the fixed-rate mortgage—but the payment for the ARM could change at any time.

\[
\text{TRY IT YOURSELF 4.18}
\]

We want to borrow \$150,000 for a 30-year home mortgage. We have found an APR of 6% for a fixed-rate mortgage and an APR of 5.7% for an ARM. Compare the monthly payments for these loans.

*The answer is provided at the end of this section.*

Example 4.18 illustrates why an ARM may seem attractive. The following example illustrates one potential danger of an ARM. It is typical of what happened in 2007 to many families who took out loans in 2006 when interest rates fell to historic lows.

\[
\text{EXAMPLE 4.19 Using an ARM: effect of increasing rates}
\]

Suppose a family has an annual income of \$60,000. Assume that this family secures a 30-year ARM for \$250,000 at an initial APR of 4.5%.

a. Find their monthly payment and the percentage of their monthly income used for the mortgage payment.
b. Now suppose that after one year the rate adjusts to 6%. Find their new monthly payment and the percentage of their monthly income used now for the mortgage payment.

Solution:

a. The monthly rate as a decimal is $0.045/12 = 0.00375$, so with $t = 360$ the monthly payment on their home is

$$\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{((1 + r)^t - 1)}$$

$$= \frac{250,000 \times 0.00375 \times 1.00375^{360}}{(1.00375^{360} - 1)}$$

$$= \$1266.71.$$ 

An annual income of $60,000 is a monthly income of $5000. To find the percentage of their monthly income used for the mortgage payment, we calculate $1266.71/5000$. The result is about 0.25, so the family is paying 25% of its monthly income for housing.

b. The APR has increased to 6%, so $r = 0.06/12 = 0.005$. To find the new monthly payment we use a loan period of 29 years or $29 \times 12 = 348$ months. For the principal we use the balance owed the bank after 1 year of payments. We noted earlier in this section that home owners build negligible equity in the first year of payments, so we will get a very good estimate of the new monthly payment if we use $250,000 as the balance owed the bank. The formula gives

$$\frac{250,000 \times 0.005 \times 1.005^{348}}{(1.005^{348} - 1)} = \$1517.51.$$ 

The family is now paying about $1517.51 - 1266.71$ or about $250 more per month, and the fraction of monthly income used for housing is about $1517.51/5000$ or about 0.30. The family is now paying 30% of its monthly income for housing.

In the example, the extra burden on family finances caused by the increase in the interest rate could lead to serious problems. Further rate adjustments could easily result in the loss of the family home.
Estimating payments on long-term loans

The payment on a long-term loan is at least as large as the monthly interest on the original balance. This is a pretty good estimate for mortgages in times of moderate-to-high interest rates.⁴

RULE OF THUMB 4.3: MONTHLY PAYMENTS FOR LONG-TERM LOANS

For all loans the monthly payment is at least as large as the principal times the monthly interest rate as a decimal. This is a fairly good estimate for a long-term loan with a moderate or high interest rate.

Let’s see what this rule of thumb would estimate for the payment on a mortgage of $100,000 at an APR of 7.2%. The monthly rate is \( \frac{7.2\%}{12} = 0.6\% \). The monthly interest on the original balance is 0.6% of $100,000:

\[
\text{Monthly payment estimate} = 100,000 \times 0.006 = 600.
\]

If this is a 30-year mortgage, the monthly payment formula gives the value $678.79 for the actual payment. This is about 13% higher than the estimate.

We should note that most home mortgage payments also include taxes and insurance. These vary from location to location but can be significant.⁵ For many, a home mortgage is the most significant investment they will ever make. It is crucial to understand clearly both the benefits and the costs of such an investment.

Derivation of the monthly payment formula

First we derive a formula for the balance still owed on an installment loan after a given number of payments.

⁴If the term is at least 30 years and the APR is at least 6% then the actual payment is within 20% of the estimate. If the term is at least 30 years and the APR is greater than 8% then the actual payment is within 10% of the estimate.

⁵See the group of exercises on affordability beginning with Exercise 39.
Suppose we borrow \( B_0 \) dollars and make monthly payments of \( M \) dollars. Suppose further that interest accrues on the balance at a monthly rate of \( r \) as a decimal. Let \( B_n \) denote the account balance (in dollars) after \( n \) months. Our goal is to derive the formula

\[
B_n = B_0 (1 + r)^n - M \frac{(1 + r)^n - 1}{r}.
\]

Each month we find the new balance \( B_{n+1} \) from the old balance \( B_n \) by first adding the interest accrued \( rB_n \) and then subtracting the payment \( M \). As a formula this is

\[
B_{n+1} = B_n + rB_n - M = B_n (1 + r) - M.
\]

If we put \( R = 1 + r \), this formula can be written more compactly as

\[
B_{n+1} = B_n R - M.
\]

Repeated application of this formula gives

\[
B_n = B_0 R^n - M (1 + R + R^2 + \cdots + R^{n-1}).
\]

To finish we use the geometric sum formula, which tells us that

\[
1 + R + R^2 + \cdots + R^{n-1} = \frac{R^n - 1}{R - 1}.
\]

This gives

\[
B_n = B_0 R^n - M \frac{R^n - 1}{R - 1}.
\]

Finally, we recall that \( R = 1 + r \) and obtain the desired formula

\[
B_n = B_0 (1 + r)^n - M \frac{(1 + r)^n - 1}{r}.
\]

Now we use the account balance formula to derive the monthly payment formula.
ALGEBRAIC SPOTLIGHT 4.3
Monthly payment formula

Suppose we borrow $B_0$ dollars at a monthly interest rate of $r$ as a decimal and we want to pay off the loan in $t$ monthly payments. That is, we want the balance to be 0 after $t$ payments. Using the account balance formula we derived in Algebraic Spotlight 4.2, we want to find the monthly payment $M$ that makes $B_t = 0$. That is, we need to solve the equation

$$0 = B_t = B_0(1 + r)^t - M \frac{(1 + r)^t - 1}{r}$$

for $M$. Now

$$0 = B_0(1 + r)^t - M \frac{(1 + r)^t - 1}{r}$$

$$M \frac{(1 + r)^t - 1}{r} = B_0(1 + r)^t$$

$$M = \frac{B_0r(1 + r)^t}{((1 + r)^t - 1)}.$$

Because $B_0$ is the amount borrowed and $t$ is the term, this is the monthly payment formula we stated earlier.

Try It Yourself Answers

TRY IT YOURSELF 4.9 Using the monthly payment formula: college loan: $199.08

TRY IT YOURSELF 4.10 Computing how much I can borrow: buying a car: $13,286.65

TRY IT YOURSELF 4.13 Estimating monthly payment: can we afford it?: No: The monthly payment is at least $500.

TRY IT YOURSELF 4.14 Making an amortization table: buying a computer:

<table>
<thead>
<tr>
<th>Payment number</th>
<th>Payment</th>
<th>Applied to interest</th>
<th>Applied to balance owed</th>
<th>Outstanding balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$55.91</td>
<td>2.5% of $1000.00 = $25.00</td>
<td>$30.91</td>
<td>$1000.00</td>
</tr>
<tr>
<td>2</td>
<td>$55.91</td>
<td>2.5% of $969.09 = $24.23</td>
<td>$31.68</td>
<td>$937.41</td>
</tr>
<tr>
<td>3</td>
<td>$55.91</td>
<td>2.5% of $937.41 = $23.44</td>
<td>$32.47</td>
<td>$904.94</td>
</tr>
</tbody>
</table>

TRY IT YOURSELF 4.16 Computing how much interest: 30-year mortgage: $521,445
TRY IT YOURSELF 4.18 Comparing monthly payments: fixed-rate and adjustable-rate mortgages: The monthly payment of $870.60 for the ARM is almost $30 less than the payment of $899.33 for the fixed-rate mortgage.

Exercise Set 4.2

Rounding in the calculation of monthly interest rates is discouraged. Such rounding can lead to answers different from those presented here. For long-term loans, the differences may be pronounced.

1. **Car payment:** To buy a car you borrow $20,000 with a term of 5 years at an APR of 6%. What is your monthly payment? How much total interest is paid?

2. **Truck payment:** You borrow $18,000 with a term of 4 years at an APR of 5% to buy a truck. What is your monthly payment? How much total interest is paid?

3. **Estimating the payment:** You borrow $25,000 with a term of 2 years at an APR of 5%. Use Rule of Thumb 4.2 to estimate your monthly payment, and compare this estimate with what the monthly payment formula gives.

4. **Estimating a mortgage:** You have a home mortgage of $110,000 with a term of 30 years at an APR of 9%. Use Rule of Thumb 4.3 to estimate your monthly payment, and compare this estimate with what the monthly payment formula gives.

5. **Affording a car:** You can get a car loan with a term of 3 years at an APR of 5%. If you can afford a monthly payment of $450, how much can you borrow?

6. **Affording a home:** You find that the going rate for a home mortgage with a term of 30 years is 6.5% APR. The lending agency says that based on your income your monthly payment can be at most $750. How much can you borrow?

7. **No interest:** A car dealer offers you a loan with no interest charged for a term of 2 years. If you need to borrow $18,000, what will your monthly payment be? Which rule of thumb is relevant here?
Interest paid: For Exercises 8 through 11 assume you take out a $2000 loan for 30 months at 8.5% APR.

8. What is the monthly payment?

9. How much of the first month’s payment is interest?

10. What percentage of the first month’s payment is interest? (Round your answer to two decimal places as a percentage.)

11. How much total interest did you pay at the end of the 30 months?

More saving and borrowing: In Exercises 12 through 14 we compare saving and borrowing as in Example 4.12.

12. Suppose you borrow $10,000 for two years at an APR of 8.75%. What will your monthly payment be? How much interest will you have paid by the end of the loan?

13. Suppose you buy a 2-year $10,000 CD at an APR of 8.75% compounded monthly. How much interest will you be paid by the end of the period?

14. In Exercise 12 the financial institution loaned you $10,000 for two years, but in Exercise 13 you loaned the financial institution $10,000 for two years. What is the difference in the amount of interest paid?

Down payment: In Exercises 15 through 17 we examine the benefits of making a down payment.

15. You want to buy a car. Suppose you borrow $15,000 for two years at an APR of 6%. What will your monthly payment be? How much interest will you have paid by the end of the loan?

16. Suppose that in the situation of Exercise 15 you make a down payment of $2000. This means that you borrow only $13,000. Assume that the term is still two years at an APR of 6%. What will your monthly payment be? How much interest will you have paid by the end of the loan?
17. Compare your answers to Exercises 15 and 16. What are the advantages of making a down payment? Why would a borrower not make a down payment? *Note:* One other factor to consider is the interest one would have earned on the down payment if it had been invested. Typically the interest rate earned on investments is lower than that charged for loans.

18. **Amortization table:** Suppose we borrow $1500 at 4% APR and pay it off in 24 monthly payments. Make an amortization table showing payments over the first 3 months.

<table>
<thead>
<tr>
<th>Payment number</th>
<th>Payment</th>
<th>Applied to interest</th>
<th>Applied to balance owed</th>
<th>Outstanding balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>$1500.00</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

19. **Another amortization table:** Suppose we borrow $100 at 5% APR and pay it off in 12 monthly payments. Make an amortization table showing payments over the first 3 months.

<table>
<thead>
<tr>
<th>Payment number</th>
<th>Payment</th>
<th>Applied to interest</th>
<th>Applied to balance owed</th>
<th>Outstanding balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>$100.00</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Term of mortgage:** There are two common choices for the term of a home mortgage: 15 years or 30 years. Suppose you need to borrow $90,000 at an annual interest rate of 6.75% to buy a home. Exercises 20 through 25 explore mortgage options for the term.

20. What will your monthly payments be if you opt for a 15-year mortgage?

21. What percentage of your first month's payment will be interest if you opt for a 15-year mortgage? (Round your answer to two decimal places as a percentage.)
22. How much interest will you have paid by the end of the 15-year loan?

23. What will your monthly payments be if you opt for a 30-year mortgage?

24. What percentage of your first month’s payment will be interest if you opt for a 30-year mortgage? (Round your answer to two decimal places as a percentage.)

25. How much interest will you have paid by the end of the 30-year loan? Is it twice as much as for a 15-year mortgage?

**Formula for equity:** Here is a formula for the equity built up after \( k \) monthly payments:

\[
\text{Equity} = \frac{\text{Amount borrowed} \times ((1 + r)^k - 1)}{((1 + r)^t - 1)},
\]

where \( r \) the monthly interest rate as a decimal and \( t \) the term in months. Exercises 26 and 27 use this formula for a mortgage of $100,000 at an APR of 7.2% with two different terms.

26. Assume that the term of the mortgage is 30 years. How much equity will you have halfway through the term of the loan? What percentage of the principal is this? (Round your answer to one decimal place as a percentage.)

27. Suppose now that instead of a 30-year mortgage you have a 15-year mortgage. Find your equity halfway through the term of the loan, and find what percentage of the principal that is. (Round your answer to one decimal place as a percentage.) Compare this with the percentage you found in Exercise 26. Why should the percent for the 15-year term be larger than the percent for the 30-year term?

**Car totaled:** In order to buy a new car you finance $20,000 with no down payment for a term of 5 years at an APR of 6%. After you have the car for 1 year you are in an accident. No one is injured, but the car is totaled. The insurance company says that before the accident the value of the car had decreased by 25% over the time you owned it, and the company pays you that depreciated amount after subtracting your $500 deductible. Exercises 28 through 31 refer to this situation.

28. What is your monthly payment for this loan?
29. How much equity have you built up after 1 year? *Suggestion:* Use the formula for equity stated in connection with Exercises 26 and 27.

30. How much money does the insurance company pay you? (Don’t forget to subtract the deductible.)

31. Can you pay off the loan using the insurance payment, or do you still need to make payments on a car you no longer have? If you still need to make payments, how much do you still owe?

**Rebates:** When interest rates are low some automobile dealers offer loans at 0% APR, as the following excerpt from an article at AutoLoanDaily.com shows.

**Chrysler Offers 0% Financing or $4,000 Consumer Cash in June (excerpt)**

By Liz Opsitnik Wednesday, Jun 03 2009 09:42. Chrysler is ramping up its June incentives by offering 0% financing for 60 months for auto loans through GMAC Financial Services, Chryslers new lender. The zero interest incentive is good on select 2009 vehicles.

Chrysler, Dodge and Jeep car shoppers can either choose the 0% financing incentive or up to $4,000 Consumer Cash on 2009 vehicles.

"Zero percent financing" means the obvious thing—that no interest is being charged on the loan. So if we borrow $1200 at 0% interest and pay it off over 12 months, our payments will be $1200/12 = $100 per month.

Suppose you are buying a new truck at a price of $20,000. You plan to finance your purchase with a loan you will repay over 2 years. The dealer offers two options: either dealer financing with 0 percent interest, or a $2000 rebate on the purchase price. If you take the rebate you will have to go to the local bank for a loan (of $18,000) at an APR of 6.5%. Exercises 32 through 34 refer to these options.

32. What would your monthly payment be if you used dealer financing?

33. What would your monthly payment be if you took the rebate?

34. Should you take the dealer financing or the rebate? How much would you save over the life of the loan by taking the option you chose?
35. **Too good to be true?** A friend claims to have found a really great deal at a local loan agency not listed in the phone book: The agency claims that its rates are so low that you can borrow $10,000 with a term of 3 years for a monthly payment of $200. Is this too good to be true? Be sure to explain your answer.

36. **Is this reasonable?** A lending agency advertises in the paper an APR of 12% on a home mortgage with a term of 30 years. The ad claims that the monthly payment on a principal of $100,000 will be $10,290. Is this claim reasonable? What should the ad have said the payment would be (to the nearest dollar)? What do you think happened here?

37. **Question for thought:** Why would it be unwise (even if it were allowed) to charge the purchase of a home to a credit card? Go beyond the credit limit to consider factors such as interest rates, budgeting, etc.

38. **Microloans and flat interest:** The 2006 Nobel Peace Prize was awarded to Muhammad Yunus and the Grameen Bank he founded. The announcement of the award noted the development of microloans for encouraging the poor to become entrepreneurs. Microloans involve small amounts of money but fairly high interest rates. In contrast to the installment method discussed in this section, for microloans typically interest is paid at a flat rate: The amount of interest paid is not reduced as the principal is paid off. Investigate flat rate loans, the reasons why microloans are often structured this way, and any downsides to such an arrangement.

**Exercises 39 through 44 are suitable for group work.**

**Affordability:** Over the past 30 years interest rates have varied widely. The rate for a 30-year mortgage reached a high of 14.75% in July of 1984, and it reached a low of 5.25% in December of 2009. A significant impact of lower interest rates on society is that they enable more people to afford the purchase of a home. In this exercise we consider the purchase of a home that sells for $125,000. Assume that we can make a down payment of $25,000, so we need to borrow $100,000. We assume that our annual income is $40,000 and that we have no other debt. In Exercises 39 through 44 we determine whether we can afford to buy the home at the high and low rates mentioned above.
39. What is our monthly income?

40. Lending agencies usually require that no more than 28% of the borrower’s monthly income be spent on housing. How much does that represent in our case?

41. The amount we will spend on housing consists of our monthly mortgage payment plus property taxes and hazard insurance. Assume that property taxes plus insurance total $250 per month, and subtract this from the answer to Exercise 40 to determine what monthly payment we can afford.

42. Use your answer to Exercise 41 to determine how much we can borrow if the term is 30 years and the interest rate is the historic high of 14.75%. Can we afford the home?

43. Use your answer to Exercise 41 to determine how much we can borrow if the term is 30 years and the interest rate is the historic low of 5.25%. Can we afford the home now?

44. What is the difference in the amount we can borrow between the lowest and highest rates?

45. **Some history:** In the United States before the 1930s home ownership was not standard—most people rented. In part this was because home loans were structured differently: A large down payment was required, the term of the loan was 5 years or less, the regular payments went toward interest, and the principal was paid off in a lump sum at the end of the term. Home mortgages as we know them came into being through the influence of the Federal Housing Administration, established by Congress in the National Housing Act in 1934, which provided insurance to lending agencies. Find more details on the history of home mortgages, and discuss why the earlier structure of loans would discourage home ownership. Are there advantages of the earlier structure?
46. **Equity**: You borrow $15,000 with a term of 4 years at an APR of 8%. Make an amortization table. How much equity have you built up halfway through the term?

47. **Finding the term**: You want to borrow $15,000 at an APR of 7% to buy a car, and you can afford a monthly payment of $500. To minimize the amount of interest paid you want to take the shortest term you can. What is the shortest term you can afford? *Note*: Your answer should be a whole number of years. Rule of Thumb 4.2 should give you a rough idea of what the term will be.
4.3 SAVING FOR THE LONG TERM: BUILD THAT NEST EGG

The following article appears on the Bankrate.com website.

**NEWSPAPER ARTICLE 4.3: RETIREMENT PLANNING FOR 20-SOMETHINGS**

**LESLIE HAGGIN GEARY**

**APRIL 23, 2007**

It’s easy to understand why retirement doesn’t loom large on the horizon for 20-somethings. Young workers are more concerned with kick-starting careers, not ending them in the long-distant future.

But it’s worth noting that the very fact that you’re young gives you a huge edge if you want to be rich in retirement…

Consider this scenario: If you begin saving for retirement at 25, putting away $2000 a year for just 40 years, you’ll have around $560,000, assuming earnings grow at 8 percent annually. Now, let’s say you wait until you’re 35 to start saving. You put away the same $2000 a year, but for three decades instead, and earnings grow at 8 percent a year. When you’re 65 you’ll wind up with around $245,000—less than half the money.

Seems like a no-brainer, right? Save a little now and reap big rewards later.

Unfortunately, many of today’s youngest workers pass on the opportunity to save for retirement early, when the beauty of compounding interest can work its magic and maximize savings. A recent study by human resources consultant Hewitt Associates found that just 31 percent of Generation Y workers (those born in 1978 or later, now in the thick of their 20s) who are eligible to put money into a 401(k) retirement savings plan to do so. That’s less than half of the 63 percent of workers between ages 26 and 41 who do invest in employer-sponsored savings accounts.

“These years of saving in your early 20s are your prime years. If you deny yourself the opportunity, it will just set you back with retirement planning in the long run,” says [one author]. “You’ve got to have balance.”

In Section 4.1 we discussed how an account grows if a lump sum is invested. Many long-term savings plans such as retirement accounts combine the growth power of compound interest with that of regular contributions. Such plans can show truly remarkable growth, the article above indicates. We look at such accounts in this sec-
Saving regular amounts monthly

Let’s look at a plan where you deposit $100 to your savings account at the end of each month, and let’s suppose the account pays you a monthly rate of 1% on the balance in the account. (That is an APR of 12% compounded monthly.)

At the end of the first month the balance is $100. At the end of the second month the account is increased by two factors—interest earned and a second deposit. Interest earned is 1% of $100 or $1.00, and the deposit is $100. This gives the new balance as

\[
\text{New balance} = \text{Previous balance} + \text{Interest} + \text{Deposit}
\]

\[
= $100 + $1 + $100 = $201.
\]

Table 4.2 tracks the growth of this account through 10 months. Note that at the end of each month the interest is calculated on the previous balance and then $100 is added to the balance.

<table>
<thead>
<tr>
<th>Month</th>
<th>Interest paid on previous balance</th>
<th>Deposit</th>
<th>Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.00</td>
<td>$100</td>
<td>$100.00</td>
</tr>
<tr>
<td>2</td>
<td>1% of $100.00 = $1.00</td>
<td>$100</td>
<td>$201.00</td>
</tr>
<tr>
<td>3</td>
<td>1% of $201.00 = $2.01</td>
<td>$100</td>
<td>$303.01</td>
</tr>
<tr>
<td>4</td>
<td>1% of $303.01 = $3.03</td>
<td>$100</td>
<td>$406.04</td>
</tr>
<tr>
<td>5</td>
<td>1% of $406.04 = $4.06</td>
<td>$100</td>
<td>$510.10</td>
</tr>
<tr>
<td>6</td>
<td>1% of $510.10 = $5.10</td>
<td>$100</td>
<td>$615.20</td>
</tr>
<tr>
<td>7</td>
<td>1% of $615.20 = $6.15</td>
<td>$100</td>
<td>$721.35</td>
</tr>
<tr>
<td>8</td>
<td>1% of $721.35 = $7.21</td>
<td>$100</td>
<td>$828.56</td>
</tr>
<tr>
<td>9</td>
<td>1% of $828.56 = $8.29</td>
<td>$100</td>
<td>$936.85</td>
</tr>
<tr>
<td>10</td>
<td>1% of $936.85 = $9.37</td>
<td>$100</td>
<td>$1046.22</td>
</tr>
</tbody>
</table>

**EXAMPLE 4.20  Verifying a balance: regular deposits into savings**

Verify the calculation shown for month 3 of Table 4.2.

Solution:

We earn 1% interest on the previous balance of $201.00:

\[
\text{Interest} = 0.01 \times $201 = $2.01.
\]
We add this amount plus a deposit of $100 to the previous balance:

\[
\text{Balance at end of month 3} = \text{Previous balance} + \text{Interest} + \text{Deposit}
\]
\[
= 201.00 + 2.01 + 100
\]
\[
= 303.01.
\]

This agrees with the entry in Table 4.2.

---

**TRY IT YOURSELF 4.20**

Verify the calculation shown for month 4 of Table 4.2.

*The answer is provided at the end of this section.*

---

In Table 4.2, the balance after 10 months is $1046.22. There were 10 deposits made, for a total of $1000. The remaining $46.22 comes from interest.

Suppose we had deposited that entire $1000 at the beginning of the 10-month period. We use the compound interest formula (Formula 4.3 from Section 4.1) to find the balance at the end of the period. The monthly rate of 1% in decimal form is \( r = 0.01 \), so we find that our balance would have been

\[
\text{Balance after 10 months} = \text{Principal} \times (1 + r)^t
\]
\[
= 1000 \times (1 + 0.01)^{10}
\]
\[
= 1104.62.
\]

That number is larger than the balance for monthly deposits because it includes interest earned on the full $1000 over the entire 10 months. Of course, for most of us it’s easier to make monthly deposits of $100 than it is to come up with a lump sum of $1000 to invest.

These observations provide a helpful rule of thumb even though we don’t yet have a formula for the balance.
**RULE OF THUMB 4.4: REGULAR DEPOSITS**

Suppose we deposit money regularly into an account with a fixed interest rate.

1. The ending balance is at least as large as the total deposit.
2. The ending balance is less than the amount we would have if the entire deposit were invested initially and allowed to draw interest over the whole term.
3. These estimates are best over a short term.

---

**Regular deposits balance formula**

The formula for the account balance assuming regular deposits at the end of each month is as follows. In this formula, \( t \) is the number of deposits, and \( r \) is the monthly interest rate APR/12 expressed as a decimal.

**Formula (4.8): Regular deposits balance formula**

\[
\text{Balance after } t \text{ deposits} = \frac{\text{Deposit} \times ((1 + r)^t - 1)}{r}.
\]

The ending balance is often called the *future value* for this savings arrangement. A derivation of this formula is given in Algebraic Spotlight 4.4 at the end of this section.

Let’s check that this formula agrees with the balance at the end of the tenth month as found in Table 4.2. The deposit is $100 each month, and the monthly rate as a decimal is \( r = 0.01 \). We want the balance after \( t = 10 \) months. Using Formula 4.8, we find

\[
\text{Balance after 10 deposits} = \frac{\text{Deposit} \times ((1 + r)^t - 1)}{r} = \frac{100 \times (1.01^{10} - 1)}{0.01} = 1046.22.
\]

This is the same as the answer we obtained in Table 4.2.

---

*See Exercises 27 and 28 for the adjustment necessary when deposits are made at the beginning of the month.*
EXAMPLE 4.21 Using the balance formula: saving money regularly

Suppose we have a savings account earning 7% APR. We deposit $20 to the account at the end of each month for 5 years. What is the future value for this savings arrangement? That is, what is the account balance after 5 years?

Solution:

We use the regular deposits balance formula (Formula 4.8). The monthly deposit is $20, and the monthly interest rate as a decimal is

$$ r = \frac{\text{APR}}{12} = \frac{0.07}{12}. $$

The number of deposits is $t = 5 \times 12 = 60$. The regular deposits balance formula gives

$$ \text{Balance after 60 deposits} = \frac{\text{Deposit} \times ((1 + r)^t - 1)}{r} = \frac{20 \times ((1 + 0.07/12)^{60} - 1)}{(0.07/12)} = 1431.86. $$

The future value is $1431.86.

TRY IT YOURSELF 4.21

Suppose we have a savings account earning 4% APR. We deposit $40 to the account at the end of each month for 10 years. What is the future value for this savings arrangement?

*The answer is provided at the end of this section.*

Determining the savings needed

People approach savings in different ways—some are committed to depositing a certain amount of money each month into a savings plan, and others save with a specific purchase in mind. Suppose you are nearing the end of your sophomore year and plan to purchase a car when you graduate in 2 years. If the car you have your eye on will cost $20,000 when you graduate, you want to know how much you will have to save each month for the next 2 years to have $20,000 at the end.

We can rearrange the regular deposits balance formula to tell us how much we need to deposit regularly in order to achieve a goal (that is, a future value) such as
this. In the following formula \( r \) is the monthly interest rate APR/12 as a decimal, and \( t \) is the number of deposits you will make to reach your goal.

### Formula (4.9): Deposit needed formula

\[
\text{Needed deposit} = \frac{\text{Goal} \times r}{((1 + r)^t - 1)}.
\]

For example, if your goal is $20,000 to buy a car in 2 years, and if the APR is 6%, we can use this formula to find how much we need to deposit each month. The monthly rate as a decimal is

\[
r = \frac{\text{APR}}{12} = \frac{0.06}{12} = 0.005.
\]

We use \( t = 2 \times 12 = 24 \) deposits in the formula:

\[
\text{Needed deposit} = \frac{\text{Goal} \times r}{((1 + r)^t - 1)} = \frac{\$20,000 \times 0.005}{(1.005^{24} - 1)} = \$786.41.
\]

You need to deposit $786.41 each month so you can buy that $20,000 car when you graduate.

#### EXAMPLE 4.22 Computing deposit needed: saving for college

How much does your younger brother need to deposit each month into a savings account that pays 7.2% APR in order to have $10,000 when he starts college in 5 years?

**Solution:**

We want to achieve a goal of $10,000 in 5 years, so we use Formula 4.9. The monthly interest rate as a decimal is

\[
r = \frac{\text{APR}}{12} = \frac{0.072}{12} = 0.006,
\]

\[
\text{Needed deposit} = \frac{\text{Goal} \times r}{((1 + r)^t - 1)} = \frac{\$10,000 \times 0.006}{(1.006^{60} - 1)} = \$733.87.
\]

Your younger brother needs to deposit $733.87 each month so he can have $10,000 when he starts college in 5 years.
and the number of deposits is $t = 5 \times 12 = 60$:

\[
\text{Needed deposit} = \frac{\text{Goal} \times r}{((1 + r)^t - 1)}
\]

\[
= \frac{10,000 \times 0.006}{(1.006^{60}) - 1}
\]

\[
= \$138.96.
\]

He needs to deposit $138.96 each month.

---

**TRY IT YOURSELF 4.22**

How much do you need to deposit each month into a savings account that pays 9% APR in order to have $50,000 for your child to use for college in 18 years?

*The answer is provided at the end of this section.*

---

**SUMMARY 4.5: MONTHLY DEPOSITS**

Suppose we deposit a certain amount of money at the end of each month into a savings account that pays a monthly interest rate of $r = \text{APR}/12$ as a decimal. The balance in the account after $t$ months is given by the regular deposits balance formula (Formula 4.8):

\[
\text{Balance after } t \text{ deposits} = \frac{\text{Deposit} \times ((1 + r)^t - 1)}{r}.
\]

The ending balance is called the future value for this savings arrangement.

A companion formula (Formula 4.9) gives the monthly deposit necessary to achieve a given balance:

\[
\text{Needed deposit} = \frac{\text{Goal} \times r}{((1 + r)^t - 1)}.
\]

---

**Saving for retirement**

As the article at the beginning of this section points out, college students often don’t think much about retirement, but early retirement planning is important.
EXAMPLE 4.23  Finding deposit needed: retirement and varying rates

Suppose that you’d like to retire in 40 years and you want to have a future value of $500,000 in a savings account. (See the article at the beginning of this section.) Also suppose that your employer makes regular monthly deposits into your retirement account.

a. If you can expect an APR of 9% for your account, how much do you need your employer to deposit each month?

b. The formulas we have been using assume that the interest rate is constant over the period in question. Over a period of 40 years, though, interest rates can vary widely. To see what difference the interest rate can make, let’s assume a constant APR of 6% for your retirement account. How much do you need your employer to deposit each month under this assumption?

Solution:

a. We have a goal of $500,000, so we use Formula 4.9. The monthly rate as a decimal is

\[ r = \frac{\text{Monthly rate}}{12} = \frac{0.09}{12} = 0.0075. \]

The number of deposits is \( t = 40 \times 12 = 480 \), so the needed deposit is

\[
\text{Needed deposit} = \frac{\text{Goal} \times r}{((1 + r)^t - 1)} = \frac{500,000 \times 0.0075}{(1.0075^{480} - 1)} = \$106.81.
\]

b. The computation is the same as in part a except that the new monthly rate as a decimal is

\[ r = \frac{0.06}{12} = 0.005. \]

We have

\[
\text{Needed deposit} = \frac{\text{Goal} \times r}{((1 + r)^t - 1)} = \frac{500,000 \times 0.005}{(1.005^{480} - 1)} = \$251.07.
\]
Note that the decrease in the interest rate from 9% to 6% requires that the monthly deposit more than double if you are to reach the same goal. The effect of this possible variation in interest rates is one factor that makes financial planning for retirement complicated.

**Retirement income: annuities**

How much income will you need in retirement? That’s a personal matter, but we can analyze what a *nest egg* will provide.

**KEY CONCEPT**

Your *nest egg* is the balance of your retirement account at the time of retirement. The *monthly yield* is the amount you can withdraw from your retirement account each month.

Once you retire there are several ways of using your retirement funds. One method is to withdraw each month only the interest accrued over that month; the principal remains the same. Under this arrangement your nest egg will never be reduced; you’ll be living off the interest alone. An arrangement like this is called a *perpetuity* because the constant income continues indefinitely. The reader is invited to explore this arrangement further in Exercises 29 through 36.

With a perpetuity the original balance at retirement remains untouched, but if we are willing to reduce the principal each month we won’t need to start with as large a nest egg for our given monthly income. In this situation we receive a constant monthly payment, part of which represents interest and part of which represents reduction of principal. Such an arrangement is called a *fixed-term annuity* because, unlike a perpetuity, this arrangement will necessarily end after a fixed term (when we have spent the entire principal).

**KEY CONCEPT**

An *annuity* is an arrangement that withdraws both principal and interest from your nest egg. Payments end when the principal is exhausted.
An annuity works just like the installment loans we considered in Section 4.2, only someone is paying us rather than the other way around. In fact, the formula for the monthly payment applies in this situation too. We can think of the institution that holds our account at retirement as having borrowed our nest egg; it will pay us back in monthly installments over the term of the annuity. In the following formula $r$ is the monthly rate (as a decimal), and $t$ is the term (the number of months the annuity will last).

**Formula (4.10): Annuity yield formula**

Monthly annuity yield = \( \frac{\text{Nest egg} \times r(1 + r)^t}{((1 + r)^t - 1)} \).

**EXAMPLE 4.24 Finding annuity yield: 20-year annuity**

Suppose we have a nest egg of $800,000 with an APR of 6% compounded monthly. Find the monthly yield for a 20-year annuity.

**Solution:**

We use the annuity yield formula (Formula 4.10). With an APR of 6%, the monthly rate as a decimal is

\[
r = \text{Monthly rate} = \frac{\text{APR}}{12} = \frac{0.06}{12} = 0.005.
\]

The term is 20 years, so we take \( t = 20 \times 12 = 240 \) months:

\[
\text{Monthly annuity yield} = \frac{\text{Nest egg} \times r(1 + r)^t}{((1 + r)^t - 1)} = \frac{800,000 \times 0.005 \times 1.005^{240}}{(1.005^{240} - 1)} = \$5731.45.
\]

**TRY IT YOURSELF 4.24**

Suppose we have a nest egg of $1,000,000 with an APR of 6% compounded monthly. Find the monthly yield for a 25-year annuity.

The answer is provided at the end of this section.

---

We assume that the payments are made at the end of the month.
How large a nest egg is needed to achieve a desired annuity yield? We can answer this question by rearranging the annuity yield formula (Formula 4.10).

**Formula (4.11):**

\[
\text{Annuity yield goal} \times \frac{((1 + r)^t - 1)}{(r(1 + r)^t)}.
\]

Here \( r \) is the monthly rate (as a decimal), and \( t \) is the term (in months) of the annuity.

The following example shows how we use this formula.

**EXAMPLE 4.25 Finding nest egg needed for annuity: retiring on a 20-year annuity**

Suppose our retirement account pays 5% APR compounded monthly. What size nest egg do we need in order to retire with a 20-year annuity that yields $4000 per month?

**Solution:**

We want to achieve an annuity goal, so we use Formula 4.11. The monthly rate as a decimal is \( r = 0.05/12 \), and the term is \( t = 20 \times 12 = 240 \) months:

\[
\text{Nest egg needed} = \frac{\text{Annuity yield goal} \times ((1 + r)^t - 1)}{(r(1 + r)^t)}.
\]

\[
= \frac{$4000 \times ((1 + 0.05/12)^{240} - 1)}{(0.05/12)(1 + 0.05/12)^{240})}
\]

\[
= $606,101.25.
\]

**TRY IT YOURSELF 4.25**

Suppose our retirement account pays 9% APR compounded monthly. What size nest egg do we need in order to retire with a 25-year annuity that yields $5000 per month?

The answer is provided at the end of this section.

The balance at retirement (the nest egg) is called the present value of the annuity. Future value and present value depend on perspective. When you started saving, the balance at retirement was the future value. When you actually retire it becomes the present value.
The obvious question is, how many years should you plan for the annuity to last? If you set it up to last until you’re 80 and then you live until you’re 85, you’re in trouble. What a retiree often wants is to have the monthly annuity payment continue for as long as she lives. Insurance companies offer such an arrangement, and it’s called a life annuity. How does it work?

The insurance company makes a statistical estimate of the life expectancy of a customer, which is used in determining how much the company will probably have to pay out. Some customers will live longer than the estimate (and the company may lose money on them) but some will not (and the company will make money on them). The monthly income for a given principal is determined from this estimate using the formula for the present value of a fixed-term annuity.

<table>
<thead>
<tr>
<th>SUMMARY 4.6: Retiring on an Annuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>For a fixed-term annuity the formulas for monthly payments apply. Let ( r ) be the monthly interest rate (as a decimal) and ( t ) the term (in months) of the annuity.</td>
</tr>
<tr>
<td><strong>1.</strong> To find the monthly yield provided by a nest egg we use</td>
</tr>
<tr>
<td>Monthly annuity yield = ( \frac{\text{Nest egg} \times r(1 + r)^t}{((1 + r)^t - 1)} ).</td>
</tr>
<tr>
<td><strong>2.</strong> To find the nest egg needed to provide a desired income we use</td>
</tr>
<tr>
<td>Nest egg needed = ( \frac{\text{Annuity yield goal} \times ((1 + r)^t - 1)}{(r(1 + r)^t)} ).</td>
</tr>
<tr>
<td>The balance at retirement (the nest egg) is called the present value of the annuity.</td>
</tr>
</tbody>
</table>

---

\(^8\)Such calculations are made by professionals known as actuaries.
Derivation of the regular deposits balance formula

**ALGEBRAIC SPOTLIGHT 4.4**

**Regular deposits balance formula**

Suppose that at the end of each month we deposit money (the same amount each month) into an account that pays a monthly rate of $r$ as a decimal. Our goal is to derive the formula

$$\text{Balance after } t \text{ deposits} = \frac{\text{Deposit} \times ((1 + r)^t - 1)}{r}.$$  

In Algebraic Spotlight 4.2 from Section 4.2, we considered the situation where we borrow $B_0$ dollars and make monthly payments of $M$ dollars at a monthly interest rate of $r$ as a decimal. We found that the balance after $t$ payments is

$$\text{Balance after } t \text{ payments} = B_0(1 + r)^t - M \frac{(1 + r)^t - 1}{r}.$$  

Making payments of $M$ dollars per month subtracts money from the balance. A deposit adds money rather than subtracting it. The result is to add the term involving $M$ instead of subtracting it:

$$\text{Balance after } t \text{ deposits} = B_0(1 + r)^t + M \frac{(1 + r)^t - 1}{r}.$$  

Now the initial balance is zero, so $B_0 = 0$. Therefore,

$$\text{Balance after } t \text{ deposits} = M \frac{(1 + r)^t - 1}{r}.$$  

Because $M$ is the amount we deposit, we can write the result as

$$\text{Balance after } t \text{ deposits} = \frac{\text{Deposit} \times ((1 + r)^t - 1)}{r}.$$  

**Try It Yourself Answers**

**TRY IT YOURSELF 4.20** Verifying a balance: regular deposits into savings: $\$303.01 + 0.01 \times $303.01 + $100 = $406.04$

**TRY IT YOURSELF 4.21** Using the balance formula: saving money regularly: $\$5889.99$

**TRY IT YOURSELF 4.22** Computing deposit needed: saving for college: $\$93.22$
TRY IT YOURSELF 4.24 Finding annuity yield: 20-year annuity: $6443.01

TRY IT YOURSELF 4.25 Finding nest egg needed for annuity: retiring on a 20-year annuity: $595,808.11

Exercise Set 4.3

1. **Saving for a car:** You are saving to buy a car, and you deposit $200 at the end of each month for 2 years at an APR of 4.8% compounded monthly. What is the future value for this savings arrangement? That is, how much money will you have for the purchase of the car after 2 years?

2. **Saving for a down payment:** You want to save up $20,000 for a down payment on a home by making regular monthly deposits over 5 years. Take the APR to be 6%. How much money do you need to deposit each month?

3. **Planning for college:** At your child’s birth you begin contributing monthly to a college fund. The fund pays an APR of 4.8% compounded monthly. You figure your child will need $40,000 at age 18 to begin college. What monthly deposit is required?

**A table:** You have a savings account into which you invest $50 at the end of every month, and the account pays you an APR of 9% compounded monthly. Exercises 4 through 7 refer to this account.

4. Fill in the following table. (Don’t use the regular deposits balance formula.)

<table>
<thead>
<tr>
<th>At end of month number</th>
<th>Interest paid on prior balance</th>
<th>Deposit</th>
<th>Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Use the regular deposits balance formula (Formula 4.8) to determine the balance in the account at the end of 4 months. Compare this to the final balance in the table from Exercise 4.
6. Use the regular deposits balance formula to determine the balance in the account at the end of 4 years.

7. Use the regular deposits balance formula to determine the balance in the account at the end of 20 years.

Another table: You have a retirement account into which your employer invests $75 at the end of every month, and the account pays an APR of 5.25% compounded monthly. Exercises 8 through 11 refer to this account.

8. Fill in the following table. (Don’t use the regular deposits balance formula.)

<table>
<thead>
<tr>
<th>At end of month number</th>
<th>Interest paid on prior balance</th>
<th>Deposit</th>
<th>Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

9. Use the regular deposits balance formula (Formula 4.8) to determine the balance in the account at the end of 4 months. Compare this to the final balance in the table from Exercise 8.

10. Use the regular deposits balance formula to determine the balance in the account at the end of 4 years.

11. Use the regular deposits balance formula to determine the balance in the account at the end of 20 years.

Is this reasonable? Jerry calculates that if he makes a deposit of $5 each month at an APR of 4.8% then at the end of 2 years he’ll have $100. Benny says that the correct amount is $135. This information is used in Exercises 12 through 14. Rule of Thumb 4.4 should be helpful here.

12. What was the total amount deposited (ignoring interest earned)? Whose answer is ruled out by this calculation? Why?
13. Suppose the total amount deposited ($5 per month for 2 years) is instead put as a lump sum at the beginning of the 2 years as principal in an account earning an APR of 4.8%. Use the compound interest formula (with monthly compounding) from Section 4.1 to determine how much would be in the account after 2 years. Whose answer is ruled out by this calculation? Why?

14. Find the correct balance after two years.

Saving for a boat: Suppose you want to save in order to purchase a new boat. In Exercises 15 and 16, take the APR to be 7.2%.

15. If you deposit $250 each month, how much will you have toward the purchase of a boat after 3 years?

16. You want to have $13,000 toward the purchase of a boat in 3 years. How much do you need to deposit each month?

17. Fixed-term annuity: You have a 20-year annuity with a present value (that is, nest egg) of $425,000. If the APR is 7%, what is the monthly yield?

18. Life annuity: You have set up a life annuity with a present value of $350,000. If your life expectancy at retirement is 21 years, what will your monthly income be? Take the APR to be 6%.

Just a bit more: You begin working at age 25, and your employer deposits $300 per month into a retirement account that pays an APR of 6% compounded monthly. You expect to retire at age 65. Use this information in Exercises 19 and 20.

19. What will be the size of your nest egg at age 65?

20. Suppose you are allowed to contribute $100 each month in addition to your employer’s contribution. What will be the size of your nest egg at age 65? Compare this with your answer to Exercise 19.

Just a bit longer: You begin working at age 25, and your employer deposits $250 each month into a retirement account that pays an APR of 6% compounded monthly. You expect to retire at age 65. Use this information for Exercises 21 and 22.
21. What will be the size of your nest egg when you retire?

22. Suppose instead that you arranged to start the regular deposits 2 years earlier, at age twenty-three. What will be the size of your nest egg when you retire? Compare this with your answer from Exercise 21.

**Planning to retire on an annuity:** You plan to work for 40 years and then retire using a 25-year annuity. You want to arrange a retirement income of $4500 per month. You have access to an account that pays an APR of 7.2% compounded monthly. Use this information for Exercises 23 and 24.

23. What size nest egg do you need to achieve the desired monthly yield?

24. What monthly deposits are required to achieve the desired monthly yield at retirement?

**Retiring:** You want to have a monthly income of $2000 from a fixed-term annuity when you retire. Take the term of the annuity to be 20 years, and assume an APR of 6% over the period of investment covered in Exercises 25 and 26.

25. How large will your nest egg have to be at retirement to guarantee the income described above?

26. You plan to make regular deposits for 40 years to build up your savings to the level you determined in Exercise 25. How large must your monthly deposit be?

**Deposits at the beginning:** In this section we considered the case of regular deposits at the end of each month. If deposits are made at the beginning of each month then the formula is a bit different. The adjusted formula is

\[
\text{Balance after } t \text{ deposits} = \text{Deposit} \times (1 + r) \times \frac{(1 + r)^t - 1}{r}.
\]

Here \( r \) is the monthly interest rate (as a decimal), and \( t \) is the number of deposits. The extra factor of \( 1 + r \) accounts for the interest earned on the deposit over the first month after it’s made.

Suppose you deposit $200 at the beginning of each month for 5 years. Take the APR to be 7.2% for Exercises 27 and 28.
27. What is the future value? In other words, what will your account balance be at the end of the period?

28. What would be the future value if we had made the deposits at the end of each month rather than at the beginning? Explain why it is reasonable that your answer here is smaller than that from Exercise 27.

29. **Retirement income: perpetuities:** If a retirement fund is set up as a perpetuity, one withdraws each month only the interest accrued over that month; the principal remains the same. For example, suppose you have accumulated $500,000 in an account with a monthly interest rate of 0.5%. Each month you can withdraw $500,000 \times 0.005 = $2500 in interest, and the nest egg will always remain at $500,000. That is, the $500,000 perpetuity has a monthly yield of $2500. In general, the monthly yield for a perpetuity is given by the formula

\[
\text{Monthly perpetuity yield} = \text{Nest egg} \times \text{Monthly interest rate}.
\]

In this formula the monthly interest rate is expressed as a decimal.

Suppose we have a perpetuity paying an APR of 6% compounded monthly. If the value of our nest egg (that is, the present value) is $800,000, find the amount we can withdraw each month. *Note:* First find the monthly interest rate.

30. **Perpetuity yield:** Refer to Exercise 29 for background on perpetuities. You have a perpetuity with a present value (that is, nest egg) of $650,000. If the APR is 5% compounded monthly, what is your monthly income?

31. **Another perpetuity:** Refer to Exercise 29 for background on perpetuities. You have a perpetuity with a present value of $900,000. If the APR is 4% compounded monthly, what is your monthly income?

32. **Comparing annuities and perpetuities:** Refer to Exercise 29 for background on perpetuities. For 40 years you invest $200 per month at an APR of 4.8% compounded monthly, then you retire and plan to live on your retirement nest egg.

   (a) How much is in your account on retirement?

   (b) Suppose you set up your account as a perpetuity on retirement. What will your monthly income be? (Assume that the APR remains at 4.8% compounded monthly.)
(c) Suppose now you use the balance in your account for a life annuity instead of a perpetuity. If your life expectancy is 21 years, what will your monthly income be? (Again, assume that the APR remains at 4.8% compounded monthly.)

(d) Compare the total amount you invested with your total return from part c. Assume that you live 21 years after retirement.

33. **Perpetuity goal: How much do I need to retire?** Refer to Exercise 29 for background on perpetuities. Here is the formula for the nest egg needed for a desired monthly yield on a perpetuity:

\[
\text{Nest egg needed} = \frac{\text{Desired monthly yield}}{\text{Monthly interest rate}}.
\]

In this formula the monthly interest rate is expressed as a decimal.

If your retirement account pays 5% APR with monthly compounding, what present value (that is, nest egg) is required for you to retire on a perpetuity that pays $4000 per month?

34. **Desired perpetuity:** Refer to the formula in Exercise 33. You want a perpetuity with a monthly income of $3000. If the APR is 7%, what does the present value need to be?

**Planning to retire on a perpetuity:** You plan to work for 40 years and then retire using a perpetuity. You want to arrange to have a retirement income of $4500 per month. You have access to an account that pays an APR of 7.2% compounded monthly. Use this information for Exercises 35 and 36.

35. **Refer to the formula in Exercise 33.** What size nest egg do you need to achieve the desired monthly yield?

36. What monthly deposits are required to achieve the desired monthly yield at retirement?

Exercises 37 through 40 are suitable for group work.

**Starting early, starting late:** In Exercises 37 through 40 we consider the effects of starting early or late to save for retirement. Assume that each account considered has an APR of 6% compounded monthly.
37. At age 20 you realize that even a modest start on saving for retirement is important. You begin depositing $50 each month into an account. What will be the value of your nest egg when you retire at age 65?

38. Against expert advice you begin your retirement program at age 40. You plan to retire at age 65. What monthly contributions do you need to make to match the nest egg from Exercise 37?

39. Compare your answer to Exercise 38 with the monthly deposit of $50 from Exercise 37. Also compare the total amount deposited in each case.

40. Let’s return to the situation in Exercise 37: At age 20 you begin depositing $50 each month into an account. Now suppose that at age 40 you finally get a job where your employer puts $400 per month into an account. You continue your $50 deposits, so from age 40 on you have two separate accounts working for you. What will be the total value of your nest egg when you retire at age 65?

41. **Retiring without interest:** Suppose we lived in a society without interest. At age 25 you begin putting $250 per month into a cookie jar until you retire at age 65. At age 65 you begin to withdraw $2500 per month from the cookie jar. How long will your retirement fund last?

42. **History of annuities:** The origins of annuities can be traced back to the ancient Romans. Look up the history of annuities and write a report on it. Include their use in Rome, in Europe during the 17th century, in colonial America, and in modern American society.

43. **History of actuaries:** Look up the history of the actuarial profession and write a report on it. Be sure to discuss what mathematics is required to become an actuary.
The following exercises are designed to be solved using technology such as calculators or computer spreadsheets. For assistance see the technology supplement.

44. **How much?** You begin saving for retirement at age 25, and you plan to retire at age 65. You want to deposit a certain amount each month into an account that pays an APR of 6% compounded monthly. Make a table that shows the amount you must deposit each month in terms of the nest egg you desire to have when you retire. Include nest egg sizes from $100,000 to $1,000,000 in increments of $100,000.

45. **How long?** You begin working at age 25, and your employer deposits $350 each month into a retirement account that pays an APR of 6% compounded monthly. Make a table that shows the size of your nest egg in terms of the age at which you retire. Include retirement ages from 60 to 70.
The accompanying excerpt is from an article at the website of CBS News. The subject is credit card debt among students in the United States, which might sound familiar to you.

NEWSPAPER ARTICLE 4.4: BEWARE OF STUDENT CREDIT CARDS
TATIANA MORALES
SEPT. 3, 2003

NEW YORK The average undergraduate leaves school with a debt of $18,900. That’s up 66 percent from five years ago, according to a new study by loan provider Nellie Mae. A large part of this is, of course, student loans, which more and more students report needing these days, thanks to ballooning tuition bills.

However, The Early Show financial advisor Ray Martin reports, a growing part of this debt is unnecessary; it’s a result of four years of charging pizza and shoes and booze on new credit cards.

College students are prime targets for credit card companies, which set up tables on campus and entice students to sign up for new cards with promises of free T-shirts or other goodies. Unfortunately, many students eagerly apply for credit and use it unwisely.

Students double their credit card debt and triple the number of cards in their wallets between the time they arrive on campus and graduation, Nellie Mae found. Another scary finding: by the time college students reach their senior year, 31 percent carry a balance of $3000 to $7000.

Most students will receive offers for “student credit cards.” These are simply cards that companies market specifically to students. The cards typically have lower credit lines—$500 to $1000—and higher interest rates. Motley Fool found that the average rate on these cards range from 10 percent to 19.8 percent.

Twenty-seven percent of students use a credit card to help finance their education. These students wind up with significantly more credit card debt when they graduate. The Nellie Mae study found that students who charged tuition and other related expenses left school with a credit card balance of $3400. This is much higher than the average graduate’s balance of $1600.

Credit cards are convenient and useful. They allow us to travel without car-
rying large sums of cash, and they sometimes allow us to defer cash payments, even interest-free, for a short time. In fact, owning a credit card can be a necessity: Most hotels and rental car companies require customers to have a credit card. Some sources say that the average credit card debt per American is almost $8000.9

The convenience comes at a cost, however. For example, some credit cards carry an APR much higher than other kinds of consumer loans. The APR depends on various factors, including the customer’s credit rating. According to the article above, credit cards intended for students may range from 10% to 19.8%. At the time of this writing, the site http://www.indexcreditcards.com/lowaprcreditcards.html reveals cards with APRs ranging from 7.24% to 22.9%.

In this section we explore credit cards in some detail. In particular, we look at how payments are calculated, the terminology used by credit card companies, and the implications of making only the minimum required payments.

Credit card basics

Different credit cards have different conditions. Some have annual fees, and the interest rates vary. Many of them will apply no finance charges if you pay the full amount owed each month. In deciding which credit card to use it is very important that you read the fine print and get the card that is most favorable to you.

Here is a simplified description of how to calculate the finance charges: Start with the balance shown in the statement from the previous month, subtract payments you’ve made since the previous statement, and add any new purchases. That is the amount that is subject to finance charges.10 Thus, the formula for finding the amount subject to finance charges is

\[
\text{Amount subject to finance charges} = \text{Previous balance} - \text{Payments} + \text{Purchases}.
\]

The finance charge is calculated by applying the monthly interest rate (the APR divided by 12) to this amount.

Let’s look at a card that charges 21.6% APR. Suppose your previous statement showed a balance of $300, you made a payment of $100 in response to your previous statement, and you have new purchases of $50. The amount subject to finance charges

---

9Averages can be a bit misleading. We will say more about this in Chapter 6.

10The most common method of calculation actually uses the average daily balance. That is the average of the daily balances over the payment period (usually one month). This means that a pair of jeans purchased early in the month will incur more finance charges than the same pair purchased late in the month. Most experts consider this a fair way to do the calculation. See Exercises 29 and 30.
Amount subject to finance charges = Previous balance − Payments + Purchases
= $300 − $100 + $50 = $250.

Because the APR is 21.6%, we find the monthly interest rate using

\[
\text{Monthly interest rate} = \frac{\text{APR}}{12} = \frac{21.6\%}{12} = 1.8\%.
\]

The finance charge is 1.8% of $250:

\[
\text{Finance charge} = 0.018 \times $250 = $4.50.
\]

This makes your new balance

\[
\text{New balance} = \text{Amount subject to finance charges} + \text{Finance charge} = $250 + $4.50 = $254.50.
\]

Your new balance is $254.50.

**EXAMPLE 4.26  Calculating finance charges: buying clothes**

Suppose your Visa card calculates finance charges using an APR of 22.8%. Your previous statement showed a balance of $500, in response to which you made a payment of $200. You then bought $400 worth of clothes, which you charged to your Visa card. Complete the following table.

<table>
<thead>
<tr>
<th>Previous balance</th>
<th>Payments</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Month 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Solution:**

We start by entering the information given.

<table>
<thead>
<tr>
<th>Previous balance</th>
<th>Payments</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Month 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$500.00</td>
<td>$200.00</td>
<td>$400.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the first month the amount subject to finance charges is

Amount subject to finance charges = Previous balance − Payments + Purchases
= $500 − $200 + $400 = $700.

The APR is 22.8%, and we divide by 12 to get a monthly rate of 1.9%. In decimal form this calculation is $0.228/12 = 0.019$. Therefore, the finance charge on $700 is

\[
\text{Finance charge} = 0.019 \times $700 = $13.30.
\]
That gives a new balance of

\[
\text{New balance} = \text{Amount subject to finance charges} + \text{Finance charge} = 700.00 + 13.30 = 713.30.
\]

Your new balance is $713.30. Here is the completed table.

<table>
<thead>
<tr>
<th></th>
<th>Previous balance</th>
<th>Payments</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 1</td>
<td>$500.00</td>
<td>$200.00</td>
<td>$400.00</td>
<td>1.9% of $700 = $13.30</td>
<td>$713.30</td>
</tr>
</tbody>
</table>

**TRY IT YOURSELF 4.26**

This is a continuation of Example 4.26. You make a payment of $300 to reduce the $713.30 balance and then charge a TV costing $700. Complete the following table.

<table>
<thead>
<tr>
<th></th>
<th>Previous balance</th>
<th>Payments</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 1</td>
<td>$500.00</td>
<td>$200.00</td>
<td>$400.00</td>
<td>$13.30</td>
<td>$713.30</td>
</tr>
<tr>
<td>Month 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The answer is provided at the end of this section.

---

**SUMMARY 4.7: CREDIT CARD BASICS**

The formula for finding the amount subject to finance charges is

\[
\text{Amount subject to finance charges} = \text{Previous balance} - \text{Payments} + \text{Purchases}.
\]

The finance charge is calculated by applying the monthly interest rate (the APR divided by 12) to this amount. The new balance is found by using the formula

\[
\text{New balance} = \text{Amount subject to finance charges} + \text{Finance charge}.
\]

**Making only minimum payments**

Most credit cards require a minimum monthly payment that is normally a fixed percentage of your balance. We will see that if you make only this minimum payment, your balance will decrease very slowly and will follow an exponential pattern.
EXAMPLE 4.27  Finding next month’s minimum payment: one payment

We have a card with an APR of 24%. The minimum payment is 5% of the balance. Suppose we have a balance of $400 on the card. We decide to stop charging and to pay it off by making the minimum payment each month. Calculate the new balance after we have made our first minimum payment, and then calculate the minimum payment due for the next month.

Solution:

The first minimum payment is

\[
\text{Minimum payment} = 5\% \text{ of balance} = 0.05 \times 400 = 20.
\]

The amount subject to finance charges is

\[
\text{Amount subject to finance charges} = \text{Previous balance} - \text{Payments} + \text{Purchases} = 400 - 20 + 0 = 380.
\]

The monthly interest rate is the APR divided by 12. In decimal form this is \(0.24/12 = 0.02\). Therefore, the finance charge on $380 is

\[
\text{Finance charge} = 0.02 \times 380 = 7.60.
\]

That makes a new balance of

\[
\text{New balance} = \text{Amount subject to finance charges} + \text{Finance charge} = 380 + 7.60 = 387.60.
\]

The next minimum payment will be 5% of this:

\[
\text{Minimum payment} = 5\% \text{ of balance} = 0.05 \times 387.60 = 19.38.
\]

TRY IT YOURSELF 4.27

This is a continuation of Example 4.27. Calculate the new balance after we have made our second minimum payment, and then calculate the minimum payment due for the next month.

The answer is provided at the end of this section.
Let’s pursue the situation described in the previous example. The following table covers the first 4 payments if we continue making the minimum 5% payment each month:

<table>
<thead>
<tr>
<th>Month</th>
<th>Previous balance</th>
<th>Minimum payment</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$400.00</td>
<td>5% of $400.00 = $20.00</td>
<td>$0.00</td>
<td>2% of $380.00 = $7.60</td>
<td>$387.60</td>
</tr>
<tr>
<td>2</td>
<td>$387.60</td>
<td>5% of $387.60 = $19.38</td>
<td>$0.00</td>
<td>2% of $368.22 = $7.36</td>
<td>$375.58</td>
</tr>
<tr>
<td>3</td>
<td>$375.58</td>
<td>5% of $375.58 = $18.78</td>
<td>$0.00</td>
<td>2% of $356.80 = $7.14</td>
<td>$363.94</td>
</tr>
<tr>
<td>4</td>
<td>$363.94</td>
<td>5% of $363.94 = $18.20</td>
<td>$0.00</td>
<td>2% of $345.74 = $6.91</td>
<td>$352.65</td>
</tr>
</tbody>
</table>

The table shows that the $400 is not being paid off very quickly. In fact, over 4 months the decrease in the balance is only about $47 (from $400 to $352.65). If we look closely at the table we will see that the balances follow a pattern. To find the pattern, let’s look at how the balance changes in terms of percentages:

Month 1: The new balance of $387.60 is 96.9% of the initial balance of $400.
Month 2: The new balance of $375.58 is 96.9% of the Month 1 new balance of $387.60.
Month 3: The new balance of $363.94 is 96.9% of the Month 2 new balance of $375.58.
Month 4: The new balance of $352.65 is 96.9% of the Month 3 new balance of $363.94.

The balance exhibits a constant percentage change. This makes sense: Each month the balance is decreased by a constant percentage due to the minimum payment and increased by a constant percentage due to the finance charge. This pattern indicates that the balance is a decreasing exponential function. That conclusion is supported by the graph of the balance shown in Figure 4.8, which has the classic shape of exponential decay.

Figure 4.8: The balance from minimum payments is a decreasing exponential function.
Because of this exponential pattern we can find a formula for the balance on our credit card in the situation where we stop charging and pay off the balance by making the minimum payment each month.

**Formula (4.12): Minimum payment balance formula**

\[
\text{Balance after } t \text{ minimum payments} = \text{Initial balance} \times ((1 + r)(1 - m))^t.
\]

In this formula \( r \) is the monthly interest rate and \( m \) is the minimum monthly payment as a percent of the balance. Both \( r \) and \( m \) are in decimal form. You should not round off the product \((1+r)(1-m)\) when performing the calculation. We provide a derivation of this formula in Algebraic Spotlight 4.5 at the end of this section.

**EXAMPLE 4.28 Using the minimum payment balance formula: balance after 2 years**

We have a card with an APR of 20% and a minimum payment that is 4% of the balance. We have a balance of $250 on the card, and we stop charging and pay the balance off by making the minimum payment each month. Find the balance after 2 years of payments.

**Solution:**

The APR in decimal form is 0.2, so the monthly interest rate in decimal form is \( r = 0.2/12 \). To avoid rounding we leave \( r \) in this form. The minimum payment is 4% of the new balance, so we use \( m = 0.04 \). The initial balance is $250. The number of payments for 2 years is \( t = 24 \). Using the minimum payment balance formula (Formula 4.12), we find

\[
\text{Balance after 24 minimum payments} = \text{Initial balance} \times ((1 + r)(1 - m))^t
\]

\[
= \$250 \times ((1 + 0.2/12)(1 - 0.04))^{24}
\]

\[
= \$250 \times ((1 + 0.2/12)(0.96))^{24}
\]

\[
= \$139.55.
\]

**TRY IT YOURSELF 4.28**

We have a card with an APR of 22% and a minimum payment that is 5% of the balance. We have a balance of $750 on the card, and we stop charging and pay the
balance off by making the minimum payment each month. Find the balance after 3 years of payments.

*The answer is provided at the end of this section.*

We already noted that the credit card balance is not paid off quickly when we make only the minimum payment each month. The reason for this is now clear: The balance is a decreasing exponential function, and such functions typically decrease very slowly in the long run. The next example illustrates the dangers of making only the minimum monthly payment.

**EXAMPLE 4.29  Paying off your credit card balance: long repayment**

Suppose you have a balance of $10,000 on your Visa card, which has an APR of 24%. The card requires a minimum payment of 5% of the balance. You stop charging and begin making only the minimum payment until your balance is below $100.

**a.** Find a formula that gives your balance after \( t \) monthly payments.

**b.** Find your balance after 5 years of payments.

**c.** Determine how long it will take to get your balance under $100.\(^{11}\)

**d.** Suppose that instead of the minimum payment you want to make a fixed monthly payment so that your debt is clear in 2 years. How much do you pay each month?

**Solution:**

**a.** The minimum rate as a decimal is \( m = 0.05 \), and the monthly rate as a decimal is \( r = 0.24/12 = 0.02 \). The initial balance is $10,000. Using the minimum payment balance formula (Formula 4.12), we find

\[
\text{Balance after } t \text{ minimum payments} = \text{Initial balance} \times ((1 + r)(1 - m))^t
\]

\[
= $10,000 \times ((1 + 0.02)(1 - 0.05))^t
\]

\[
= $10,000 \times 0.969^t.
\]

\(^{11}\)It is worth noting that paying in this fashion will never actually get your balance to exactly 0. But doing so will eventually make the balance small enough that you could pay it off in a lump sum. Normally credit cards take care of this with a requirement such as “The minimum payment is 5% or $20, whichever is larger.”
b. Now 5 years is 60 months, so we put \( t = 60 \) into the formula from part a:

\[
\text{Balance after 60 months} = 10,000 \times 0.969^{60} = 1511.56.
\]

After 5 years we still owe over fifteen hundred dollars.

c. We will show two ways to determine how long it takes to get the balance down to $100.

Method 1: Using a logarithm: We need to solve for \( t \) the equation

\[
100 = 10,000 \times 0.969^t.
\]

The first step is to divide each side of the equation by 10,000:

\[
\frac{100}{10,000} = \frac{10,000}{10,000} \times 0.969^t
\]

\[
0.01 = 0.969^t.
\]

In Section 3.3 we learned how to solve exponential equations using logarithms. Summary 3.9 tells us that the solution for \( t \) of the equation \( A = B^t \) is

\[
t = \frac{\log A}{\log B}.
\]

Using this formula with \( A = 0.01 \) and \( B = 0.969 \) gives

\[
t = \frac{\log 0.01}{\log 0.969} \text{ months}.
\]

This is about 146.2 months. Hence, the balance will be under $100 after 147 monthly payments, or more than 12 years of payments.

Method 2: Trial and error: If you want to avoid logarithms you can solve this problem using trial and error with a calculator. The information in part b indicates that it will take some time for the balance to drop below $100. So we might try 10 years, or 120 months. Computation using the formula from part a shows that after 10 years the balance is still over $200, so we should try a larger number of months. If we continue in this way, we find the same answer as that obtained from Method 1: The balance drops below $100 at payment 147, which represents over 12 years of payments. (Spreadsheets and many calculators will create tables of values that make problems of this sort easy to solve.)

d. Making fixed monthly payments to clear your debt is like considering your debt as an installment loan: Just find the monthly payment if you borrow $10,000 to
buy (say) a car at an APR of 24% and pay the loan off over 24 months. We use the monthly payment formula from Section 4.2:

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{(1 + r)^t - 1}.
\]

Recall that \(t\) is the number of months taken to pay off the loan, in this case 24, and that \(r\) is the monthly rate as a decimal, which in this case is 0.02. Hence

\[
\text{Monthly payment} = \frac{\$10,000 \times 0.02 \times 1.02^{24}}{(1.02^{24} - 1)} = \$528.71.
\]

So a payment of $528.71 each month will clear the debt in 2 years.

---

**SUMMARY 4.8: MAKING MINIMUM PAYMENTS**

Suppose we have a balance on our credit card and decide to stop charging and pay off the balance by making the minimum payment each month.

1. The balance is given by the exponential formula

\[
\text{Balance after } t \text{ minimum payments} = \text{Initial balance} \times ((1 + r)(1 - m))^t.
\]

In this formula \(r\) is the monthly interest rate and \(m\) is the minimum monthly payment as a percent of the balance. Both \(r\) and \(m\) are in decimal form.

2. The product \((1 + r)(1 - m)\) should not be rounded when the calculation is performed.

3. Because the balance is a decreasing exponential function, the balance decreases very slowly in the long run.
Further complications of credit card usage

Situations involving credit cards can often be even more complicated than those discussed in previous examples in this section. For instance, in each of those examples there was a single purchase and no further usage of the card. Of course, it’s more common to make purchases every month. We also assumed that your payments were made on time, but there are substantial penalties for late or missed payments.

Another complication is that credit card companies sometimes have “specials” or promotions in which you are allowed to skip a payment. And then there are cash advances, which are treated differently from purchases. Typically cash advances incur finance charges immediately rather than after a month; that is, a cash advance is treated as carrying a balance immediately. Also cash advances incur higher finance charges than purchases do.

Another complication occurs when your credit limit is reached (popularly known as “maxing out your card”). The credit limit is the maximum balance the credit card company allows you to carry. Usually the limit is based on your credit history and your ability to pay. When you max out a credit card, the company will sometimes raise your credit limit, if you have always made required payments by their due dates.

In addition to all of these complications, there are often devious hidden fees and charges, as one can see in the following excerpt from an editorial in the *St. Petersburg Times.*
The credit card industry has figured out all sorts of ways to trap consumers into paying high fees and interest charges. Some of those practices that one would think would be illegal, such as retroactively raising interest rates on current balances, are fairly common and raise billions of dollars in profits from unwitting cardholders. Finally, it appears Congress will put an end to the worst elements of the credit card business.

Last month in an overwhelming and bipartisan vote, the House approved the Credit Card Holders’ Bill of Rights. It has the support of President Barack Obama, who met recently with credit card executives only to tell them that he would sign the bill. Now the Senate is expected to take up the issue this week, and it should not water down key consumer protections.

Americans owe more than $960 billion in credit card debt and should be held accountable for that spending. But unethical “gotcha” practices of the industry for late or incomplete payments exact too high a financial punishment.

For example, a common gimmick is for companies to avoid applying payments to the debt that carries the highest interest rate, a practice that the House bill would outlaw. It also would ban double-cycle billing, a particularly noxious practice in which interest is charged on prior balances that have been paid off if the consumer revolves a balance the next month. So if a consumer pays off a $600 balance one month but leaves $100 balance unpaid the next month, he is charged interest on $700.

For far too long the credit card industry has been given a free hand to pick consumers’ pockets. If passed, the Credit Card Holders’ Bill of Rights will turn industry practices that were outrageously abusive into outlawed acts; and Congress will have finally done its job to act in the public’s interest. The House has approved significant reforms, and the bipartisan effort that will kick off this week in the Senate should be just as strong.
Suppose a credit card has an initial balance. Assume that we incur no further charges and make only the minimum payment each month. Suppose \( r \) is the monthly interest rate and \( m \) is the minimum monthly payment as a percentage of the new balance. Both \( r \) and \( m \) are in decimal form. Our goal is to derive the formula

\[
\text{Balance after } t \text{ minimum payments} = \text{Initial balance} \times ((1 + r)(1 - m))^t.
\]

Here is the derivation. Assume that we have made a series of payments, and let \( B \) denote the balance remaining. We need to calculate the new balance. First we find the minimum payment on the balance \( B \). To do so we multiply \( B \) by \( m \):

\[
\text{Minimum payment} = mB.
\]

Next we find the amount subject to finance charges:

\[
\text{Amount subject to finance charges} = \text{Previous balance} - \text{Payments} = B - mB = B(1 - m).
\]

To calculate the finance charge we apply the monthly rate \( r \) to this amount:

\[
\text{Finance charge} = r \times B(1 - m).
\]

Therefore, the new balance is

\[
\text{New balance} = \text{Amount subject to finance charges} + \text{Finance charge} = B(1 - m) + rB(1 - m) = B \times ((1 - m) + r(1 - m)) = B \times ((1 + r)(1 - m)).
\]

We can write this as

\[
\text{New balance} = \text{Previous balance} \times ((1 + r)(1 - m)).
\]

Therefore, to find the new balance each month we multiply the previous balance by \((1+r)(1-m)\). That makes the balance after \( t \) payments an exponential function of \( t \) with base \((1+r)(1-m)\). The initial value of this function is the initial balance, so we have the exponential formula

\[
\text{Balance after } t \text{ minimum payments} = \text{Initial balance} \times ((1 + r)(1 - m))^t.
\]

This is the minimum payment balance formula.
Try It Yourself Answers

TRY IT YOURSELF 4.26 Calculating finance charges: buying clothes:

<table>
<thead>
<tr>
<th>Month</th>
<th>Previous balance</th>
<th>Payments</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 1</td>
<td>$500.00</td>
<td>$200.00</td>
<td>$400.00</td>
<td>$13.30</td>
<td>$713.30</td>
</tr>
<tr>
<td>Month 2</td>
<td>$713.30</td>
<td>$300.00</td>
<td>$700.00</td>
<td>$21.15</td>
<td>$1134.45</td>
</tr>
</tbody>
</table>

TRY IT YOURSELF 4.27 Finding next month’s minimum payment: one payment: New balance: $375.58; minimum payment: $18.78.

TRY IT YOURSELF 4.28 Using the minimum payment balance formula: balance after 2 years: $227.59

Exercise Set 4.4

1. Calculating balances: You have a credit card with an APR of 16%. You begin with a balance of $800, in response to which you make a payment of $400. The first month you make charges amounting to $300. You make a payment of $300 to reduce the new balance, and the second month you charge $600. Complete the following table.

<table>
<thead>
<tr>
<th>Month</th>
<th>Previous balance</th>
<th>Payments</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 1</td>
<td>$800.00</td>
<td>$400.00</td>
<td>$300.00</td>
<td>$128.00</td>
<td>$168.00</td>
</tr>
<tr>
<td>Month 2</td>
<td>$168.00</td>
<td>$300.00</td>
<td>$600.00</td>
<td>$99.20</td>
<td>$187.20</td>
</tr>
</tbody>
</table>

2. Calculating balances: You have a credit card with an APR of 20%. You begin with a balance of $600, in response to which you make a payment of $400. The first month you make charges amounting to $200. You make a payment of $300 to reduce the new balance, and the second month you charge $100. Complete the following table.

<table>
<thead>
<tr>
<th>Month</th>
<th>Previous balance</th>
<th>Payments</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 1</td>
<td>$600.00</td>
<td>$400.00</td>
<td>$200.00</td>
<td>$120.00</td>
<td>$80.00</td>
</tr>
<tr>
<td>Month 2</td>
<td>$80.00</td>
<td>$300.00</td>
<td>$100.00</td>
<td>$24.00</td>
<td>$164.00</td>
</tr>
</tbody>
</table>
3. **Calculating balances:** You have a credit card with an APR of 22.8%. You begin with a balance of $1000, in response to which you make a payment of $200. The first month you make charges amounting to $500. You make a payment of $200 to reduce the new balance, and the second month you charge $600. Complete the following table.

<table>
<thead>
<tr>
<th>Month</th>
<th>Previous balance</th>
<th>Payments</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Month 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. **Calculating balances:** You have a credit card with an APR of 12%. You begin with a balance of $200, in response to which you make a payment of $75. The first month you make charges amounting to $50. You make a payment of $75 to reduce the new balance, and the second month you charge $60. Complete the following table.

<table>
<thead>
<tr>
<th>Month</th>
<th>Previous balance</th>
<th>Payments</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Month 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. **A balance statement:** Assume you start with a balance of $4500 on your Visa credit card. During the first month you charge $500, and during the second month you charge $300. Assume that Visa has finance charges of 24% APR and that each month you make only the minimum payment of 2.5% of the balance. Complete the following table.

<table>
<thead>
<tr>
<th>Month</th>
<th>Previous balance</th>
<th>Minimum payment</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$4500.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$4500.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. **A balance statement:** Assume you start with a balance of $4500 on your Visa credit card. Assume that Visa has finance charges of 12% APR and that each month you make only the minimum payment of 3% of the balance. In the first month you charge $300, and in the second month you charge $600. Complete the following table.

<table>
<thead>
<tr>
<th>Month</th>
<th>Previous balance</th>
<th>Minimum payment</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$4500.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$4500.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
New balances: Assume that you have a balance of $4500 on your Visa credit card and that you make no more charges. Assume that Visa charges 21% APR and that each month you make only the minimum payment of 2.5% of the balance. This is the setting for Exercises 7 through 9.

7. Find a formula for the balance after $t$ monthly payments.

8. What will the balance be after 30 months?

9. What will the balance be after 10 years?

Paying tuition on your American Express card at the maximum interest rate:
You have a balance of $10,000 for your tuition on your American Express credit card. Assume that you make no more charges on the card. Also assume that American Express charges 24% APR and that each month you make only the minimum payment of 2% of the balance. This is the setting for Exercises 10 through 13.

10. Find a formula for the balance after $t$ monthly payments.

11. How much will you owe after 10 years of payments?

12. How much would you owe if you made 100 years of payments?

13. Find when the balance would be $50 or less.

Paying tuition on your American Express card:
You have a balance of $10,000 for your tuition on your American Express credit card. Assume that you make no more charges on the card. Also assume that American Express charges 12% APR and that each month you make only the minimum payment of 4% of the balance. This is the setting for Exercises 14 and 15.

14. Find a formula for the balance after $t$ monthly payments.

15. How long will it take to get the balance below $50$?
**Paying off a Visa card:** You have a balance of $1000 on your Visa credit card. Assume that you make no more charges on the card and that the card charges 9.9% APR and requires a minimum payment of 3% of the balance. Assume also that you make only the minimum payments. This is the setting for Exercises 16 and 17.

16. Find a formula for the balance after \( t \) monthly payments.

17. Find how many months it takes to bring the balance below $50.

18. **Balance below $200:** You have a balance of $4000 on your credit card, and you make no more charges. Assume the card requires a minimum payment of 5% and carries an APR of 22.8%. Assume also that you make only the minimum payments. Determine when the balance drops below $200.

19. **New balances:** Assume that you have a balance of $3000 on your Visa credit card and that you make no more charges. Assume that Visa charges 12% APR and that each month you make only the minimum payment of 5% of the balance. This is the setting for Exercises 19 through 23.

20. What will the balance be after 30 months?

21. What will the balance be after 10 years?

22. At what balance do you begin making payments of $20 or less?

23. Find how many months it will take to bring the remaining balance down to the value from Exercise 22.

**Paying off an American Express card:** Assume that you have a balance of $3000 on your American Express credit card and that you make no more charges. Assume that American Express charges 21% APR and that each month you make only the minimum payment of 2% of the balance. This is the setting for Exercises 24 through 26.
24. Find a formula for the remaining balance after \( t \) monthly payments.

25. On what balance do you begin making payments of $50 or less?

26. Find how many months it will take to bring the remaining balance down to the value from Exercise 25.

27. **Monthly payment:** You have a balance of $400 on your credit card and make no more charges. Assume the card carries an APR of 18%. Suppose you wish to pay off the card in 6 months by making equal payments each month. What is your monthly payment?

28. **What can you afford to charge?** Suppose you have a new credit card with 0% APR for a limited period. The card requires a minimum payment of 5% of the balance. You feel you can afford to pay no more than $250 each month. How much can you afford to charge? How much could you afford to charge if the minimum payment were 2% instead of 5%?

**Average daily balance:** The most common way of calculating finance charges is not the simplified one we used in this section but rather the *average daily balance*. With this method we calculate the account balance at the end of each day of the month and take the average. That average is the amount subject to finance charges. To simplify things, we assume that the billing period is one week rather than one month. Assume that the weekly rate is 1%. You begin with a balance of $500. On day 1 you charge $75. On day 3 you make a payment of $200 and charge $100. On day 6 you charge $200. Use this information for Exercises 29 and 30.

29. Assume that finance charges are calculated using the simplified method shown in this section. Find the account balance at the end of the week.

30. Assume that finance charges are calculated using the average daily balance. Find the account balance at the end of the week.
More on average daily balance: The method for calculating finance charges based on the average daily balance is explained in the setting for Exercises 29 and 30. As in those exercises, to simplify things we assume that the billing period is one week rather than one month and that the weekly rate is 1%. You begin with a balance of $1000. On day 1 you charge $200. On day 3 you charge $500. On day 6 you make a payment of $400 and you charge $100. Use this information for Exercises 31 and 32.

31. Assume that finance charges are calculated using the simplified method shown in this section. Find the account balance at the end of the week.

32. Assume that finance charges are calculated using the average daily balance. Find the account balance at the end of the week.

33. Finance charges versus minimum payments: In all of the examples in this section, the monthly finance charge is always less than the minimum payment. In fact, this is always the case. Explain what would happen if the minimum payment were less than the monthly finance charge.

34. A bit of history: Credit cards are relatively new. Write a brief report on the introduction of credit cards into the U.S. economy.

The following exercises are designed to be solved using technology such as calculators or computer spreadsheets. For assistance see the technology supplement.

35. Paying off a Visa card—in detail: Assume that you have a balance of $1000 on your Visa credit card and that you make no more charges on the card. Assume that Visa charges 12% APR and that the minimum payment is 5% of the balance each month. Assume also that you make only the minimum payments. Make a spreadsheet listing the items below for each month until the payment falls below $20.

<table>
<thead>
<tr>
<th>Month</th>
<th>Previous balance</th>
<th>Minimum payment</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1000.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$900.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>and so on</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


36. Paying off an American Express card: Assume that you have a balance of $3000 on your American Express credit card and that you make no more charges on the card. Assume that American Express charges 20.5% APR and that the minimum payment is 2% of the balance each month. Assume also that you make only the minimum payments. Make a spreadsheet listing the items below for each month until the payment drops below $50.

<table>
<thead>
<tr>
<th>Month</th>
<th>Previous balance</th>
<th>Minimum payment</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3000.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>and so on</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4.5 INFLATION, TAXES AND STOCKS: MANAGING YOUR MONEY

The following excerpt is from an article at CNNMoney.com.

**NEWSPAPER ARTICLE 4.6: BEN BERNANKE’S HIGH-WIRE ACT**

**FED CHIEF, IN FIRST OF TWO DAYS OF TESTIMONY ON CAPITOL HILL, ACKNOWLEDGES TROUBLING SIGNS ABOUT ECONOMIC GROWTH BUT ALSO RAISES CONCERNS ABOUT INFLATION.**

**BY DAVID ELLIS**

**FEBRUARY 28 2008**

WASHINGTON (CNNMoney.com) – For Federal Reserve Chairman Ben Bernanke, running the central bank has become an increasingly challenging high-wire balancing act.

All of Wall Street was watching the Fed chairman on Wednesday when he headed to Capitol Hill to outline the trio of challenges facing the Fed: an economy at risk of falling into a recession, topsy-turvy financial markets and the rising risk of inflation.

“We do face a difficult situation,” Bernanke told members of the House Financial Services Committee, marking the first day of his two-day semi-annual hearing on the Fed’s monetary policy. “The challenge for us is to balance those risks and decide at any given time which is more serious.”

... Bernanke’s comments were in line with the Fed’s latest economic outlook and remarks he delivered alongside Treasury Secretary Henry Paulson before a Senate panel nearly two weeks ago.

At the time, the two policymakers warned of slower economic growth in the coming year but said they believed the U.S. economy would avoid tipping into a recession, helped in part by the $170 billion economic stimulus package signed by President Bush on Feb. 13 and the most recent interest rate cuts by the Federal Reserve.

This article refers to financial issues such as inflation, financial markets, and a stimulus package that involved tax rebates. We will explore such issues in this section.
CPI and the inflation rate

The above article states the concerns of the Federal Reserve Board about inflation. But what is inflation, and how is it measured?

Inflation is calculated using the Consumer Price Index (CPI), which is a measure of the price of a certain “market basket” of consumer goods and services relative to a predetermined benchmark. When the CPI goes up, we have inflation. When it goes down, we have deflation.

According to the U.S. Department of Labor this “market basket” consists of commodities in the following categories:

- FOOD AND BEVERAGES (breakfast cereal, milk, coffee, chicken, wine, service meals and snacks)
- HOUSING (rent of primary residence, owners’ equivalent rent, fuel oil, bedroom furniture)
- APPAREL (men’s shirts and sweaters, women’s dresses, jewelry)
- TRANSPORTATION (new vehicles, airline fares, gasoline, motor vehicle insurance)
- MEDICAL CARE (prescription drugs and medical supplies, physicians’ services, eyeglasses and eye care, hospital services)
- RECREATION (televisions, pets and pet products, sports equipment, admissions)
- EDUCATION AND COMMUNICATION (college tuition, postage, telephone services, computer software and accessories)
- OTHER GOODS AND SERVICES (tobacco and smoking products, haircuts and other personal services, funeral expenses)

KEY CONCEPT

The Consumer Price Index (CPI) is a measure of the average price paid by urban consumers for a “market basket” of consumer goods and services.
The rate of inflation is measured by the percentage change in the CPI.

**KEY CONCEPT**

An increase in prices is referred to as inflation. The rate of inflation is measured by the percentage change in the Consumer Price Index over time. When prices decrease, the percentage change is negative; this is referred to as deflation.

Inflation reflects a decline of the purchasing power of the consumer’s dollar. Besides affecting how much we can afford to buy, the inflation rate has a big influence on certain government programs that impact the lives of many people.

For example, about 48 million Social Security beneficiaries, 20 million food stamp recipients, 4 million military and Federal Civil Service retirees and survivors, and 27 million children who eat subsidized lunches at school are affected by the CPI because these benefits are adjusted periodically to compensate for inflation.

Table 4.3 shows the annual change in prices in the U.S. over a 60-year period. For example, from December of 1945 to December of 1946 the CPI changed from 18.2 to 21.5, an increase of \( 21.5 - 18.2 = 3.3 \). Now we find the percentage change:

\[
\text{Percentage change} = \frac{\text{Change in CPI}}{\text{Previous CPI}} \times 100\% = \frac{3.3}{18.2} \times 100\%.
\]

This is about 18.1%, and that is the inflation rate for this period shown in the table. Usually we round the inflation rate as a percentage to 1 decimal place.
### Table 4.3
**Historical inflation**

<table>
<thead>
<tr>
<th>December</th>
<th>CPI</th>
<th>Inflation rate</th>
<th>December</th>
<th>CPI</th>
<th>Inflation rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1945</td>
<td>18.2</td>
<td>—</td>
<td>1975</td>
<td>55.5</td>
<td>6.9%</td>
</tr>
<tr>
<td>1946</td>
<td>21.5</td>
<td>18.1%</td>
<td>1976</td>
<td>58.2</td>
<td>4.9%</td>
</tr>
<tr>
<td>1947</td>
<td>23.4</td>
<td>8.8%</td>
<td>1977</td>
<td>62.1</td>
<td>6.7%</td>
</tr>
<tr>
<td>1948</td>
<td>24.1</td>
<td>3.0%</td>
<td>1978</td>
<td>67.7</td>
<td>9.0%</td>
</tr>
<tr>
<td>1949</td>
<td>23.6</td>
<td>−2.1%</td>
<td>1979</td>
<td>76.7</td>
<td>13.3%</td>
</tr>
<tr>
<td>1950</td>
<td>25.0</td>
<td>5.9%</td>
<td>1980</td>
<td>86.3</td>
<td>12.5%</td>
</tr>
<tr>
<td>1951</td>
<td>26.5</td>
<td>6.0%</td>
<td>1981</td>
<td>94.0</td>
<td>8.9%</td>
</tr>
<tr>
<td>1952</td>
<td>26.7</td>
<td>0.8%</td>
<td>1982</td>
<td>97.6</td>
<td>3.8%</td>
</tr>
<tr>
<td>1953</td>
<td>26.9</td>
<td>0.7%</td>
<td>1983</td>
<td>101.3</td>
<td>3.8%</td>
</tr>
<tr>
<td>1954</td>
<td>26.7</td>
<td>−0.7%</td>
<td>1984</td>
<td>105.3</td>
<td>3.9%</td>
</tr>
<tr>
<td>1955</td>
<td>26.8</td>
<td>0.4%</td>
<td>1985</td>
<td>109.3</td>
<td>3.8%</td>
</tr>
<tr>
<td>1956</td>
<td>27.6</td>
<td>3.0%</td>
<td>1986</td>
<td>110.5</td>
<td>1.1%</td>
</tr>
<tr>
<td>1957</td>
<td>28.4</td>
<td>2.9%</td>
<td>1987</td>
<td>115.4</td>
<td>4.4%</td>
</tr>
<tr>
<td>1958</td>
<td>28.9</td>
<td>1.8%</td>
<td>1988</td>
<td>120.5</td>
<td>4.4%</td>
</tr>
<tr>
<td>1959</td>
<td>29.4</td>
<td>1.7%</td>
<td>1989</td>
<td>126.1</td>
<td>4.6%</td>
</tr>
<tr>
<td>1960</td>
<td>29.8</td>
<td>1.4%</td>
<td>1990</td>
<td>133.8</td>
<td>6.1%</td>
</tr>
<tr>
<td>1961</td>
<td>30.0</td>
<td>0.7%</td>
<td>1991</td>
<td>137.9</td>
<td>3.1%</td>
</tr>
<tr>
<td>1962</td>
<td>30.4</td>
<td>1.3%</td>
<td>1992</td>
<td>141.9</td>
<td>2.9%</td>
</tr>
<tr>
<td>1963</td>
<td>30.9</td>
<td>1.6%</td>
<td>1993</td>
<td>145.8</td>
<td>2.7%</td>
</tr>
<tr>
<td>1964</td>
<td>31.2</td>
<td>1.0%</td>
<td>1994</td>
<td>149.7</td>
<td>2.7%</td>
</tr>
<tr>
<td>1965</td>
<td>31.8</td>
<td>1.9%</td>
<td>1995</td>
<td>153.5</td>
<td>2.5%</td>
</tr>
<tr>
<td>1966</td>
<td>32.9</td>
<td>3.5%</td>
<td>1996</td>
<td>158.6</td>
<td>3.3%</td>
</tr>
<tr>
<td>1967</td>
<td>33.9</td>
<td>3.0%</td>
<td>1997</td>
<td>161.3</td>
<td>1.7%</td>
</tr>
<tr>
<td>1968</td>
<td>35.5</td>
<td>4.7%</td>
<td>1998</td>
<td>163.9</td>
<td>1.6%</td>
</tr>
<tr>
<td>1969</td>
<td>37.7</td>
<td>6.2%</td>
<td>1999</td>
<td>168.3</td>
<td>2.7%</td>
</tr>
<tr>
<td>1970</td>
<td>39.8</td>
<td>5.6%</td>
<td>2000</td>
<td>174.0</td>
<td>3.4%</td>
</tr>
<tr>
<td>1971</td>
<td>41.1</td>
<td>3.3%</td>
<td>2001</td>
<td>176.7</td>
<td>1.6%</td>
</tr>
<tr>
<td>1972</td>
<td>42.5</td>
<td>3.4%</td>
<td>2002</td>
<td>180.9</td>
<td>2.4%</td>
</tr>
<tr>
<td>1973</td>
<td>46.2</td>
<td>8.7%</td>
<td>2003</td>
<td>184.3</td>
<td>1.9%</td>
</tr>
<tr>
<td>1974</td>
<td>51.9</td>
<td>12.3%</td>
<td>2004</td>
<td>190.3</td>
<td>3.3%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2005</td>
<td>196.8</td>
<td>3.4%</td>
</tr>
</tbody>
</table>

**Example 4.30 Calculating inflation: CPI increase to 205**

Suppose the CPI increases this year from 200 to 205. What is the rate of inflation for this year?

**Solution:**

The change in the CPI is $205 - 200 = 5$. To find the percentage change, we divide the increase of 5 by the original value of 200 and convert to a percent:

$$\text{Percentage change} = \frac{\text{Change in CPI}}{\text{Previous CPI}} \times 100\% = \frac{5}{200} \times 100\% = 2.5\%.$$
Therefore, the rate of inflation is 2.5%.

TRY IT YOURSELF 4.30
Suppose the CPI increases this year from 215 to 225. What is the rate of inflation for this year?
The answer is provided at the end of this section.

Some countries have experienced very high rates of inflation, sometimes referred to as hyperinflation. Here is a table of the five countries with the highest rates of inflation for 2008. Note that the inflation rate in Zimbabwe for 2008 is estimated to be 11.2 million percent.

Table 4.4
Examples of hyperinflation in 2008

<table>
<thead>
<tr>
<th>Country</th>
<th>Estimated inflation rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zimbabwe</td>
<td>11,200,000%</td>
</tr>
<tr>
<td>Ethiopia</td>
<td>41%</td>
</tr>
<tr>
<td>Venezuela</td>
<td>31%</td>
</tr>
<tr>
<td>Guinea</td>
<td>30%</td>
</tr>
<tr>
<td>Mongolia</td>
<td>28%</td>
</tr>
</tbody>
</table>

If the rate of inflation is 10% one may think that the buying power of a dollar has decreased by 10%, but that is not the case. Inflation is the percentage change in prices, and that is not the same as the percentage change in the value of a dollar. To see the difference, let’s imagine a frightening inflation rate this year of 100%. With such a rate, an item that costs $200 this year will cost $400 next year. This means that my money can buy only half as much next year as it can this year. So the buying power of a dollar would decrease by 50%, not by 100%.

The following formula tells us how much the buying power of currency decreases for a given inflation rate.

\[
\text{Percent decrease in buying power} = \frac{100i}{100 + i}.
\]

\(^{12}\text{From the World Factbook of the CIA.}\)
Here \( i \) is the inflation rate expressed as a percent, not a decimal. Usually we round the decrease in buying power as a percentage to 1 decimal place.

The buying power formula is derived in Algebraic Spotlight 4.6 at the end of this section.

**EXAMPLE 4.31  Calculating decrease in buying power: 5% inflation**

Suppose the rate of inflation this year is 5%. What is the percentage decrease in the buying power of a dollar?

**Solution:**

We use the buying power formula (Formula 4.13) with \( i = 5\% \):

\[
\text{Percent decrease in buying power} = \frac{100 \times i}{100 + i} = \frac{100 \times 5}{100 + 5}.
\]

This is about 4.8%.

**TRY IT YOURSELF 4.31**

According to Table 4.4, in 2008 the rate of inflation for Venezuela was 31\%. What was the percentage decrease that year in the buying power of the bolívar (the currency of Venezuela)?

*The answer is provided at the end of this section.*

A companion formula to the buying power formula (Formula 4.13) gives the inflation rate in terms of the percent decrease in buying power of currency.

**Formula (4.14): Inflation formula**

\[
\text{Percent rate of inflation} = \frac{100B}{100 - B}.
\]

In this formula, \( B \) is the decrease in buying power expressed as a percent, not as a decimal.
EXAMPLE 4.32  Calculating inflation: 2.5% decrease in buying power

Suppose the buying power of a dollar decreases by 2.5% this year. What is the rate of inflation this year?

Solution:

We use the inflation formula (Formula 4.14) with \( B = 2.5\% \):

\[
\text{Percent rate of inflation} = \frac{100B}{100 - B} = \frac{100 \times 2.5}{100 - 2.5}.
\]

This is about 2.6%.

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TRY IT YOURSELF 4.32

Suppose the buying power of a dollar decreases by 5.2% this year. What is the rate of inflation this year?

The answer is provided at the end of this section.

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The next example covers all of the concepts we have considered so far in this section.

EXAMPLE 4.33  Understanding inflation and buying power: effects on goods

Parts a through c refer to Table 4.3.


b. If a sofa cost $100 in December of 1966 and the price changed in accordance with the inflation rate in the table, how much did the sofa cost in December of 1967?

c. If a chair cost $50 in December of 1948 and the price changed in accordance with the inflation rate in the table, how much did the chair cost in December of 1949?

d. According to Table 4.4, in 2008 the rate of inflation for Ethiopia was 41%. How much did the buying power of the currency, the birr, decrease during the year?

e. The inflation rate in Ukraine for 2008 was 25.0%. In Kenya the buying power of the currency decreased by 20.3% in 2008. Which of these two countries had the larger inflation rate for 2008?
Solution:

a. In 1990 the CPI was 133.8, and in 2000 it was 174.0. The increase was $174.0 - 133.8 = 40.2$, so

\[
\text{Percentage change} = \frac{\text{Change in CPI}}{\text{Previous CPI}} \times 100\% = \frac{40.2}{133.8} \times 100\%
\]

or about 30.0%. The 10-year inflation rate was about 30.0%.

b. According to the table, the inflation rate was 3% in 1967, which tells us that the price of the sofa increased by 3% during that year. Therefore it cost $100 + 0.03 \times 100 = \$103$ in December of 1967.

c. The inflation rate is $-2.1\%$, which tells us that the price of the chair decreased by 2.1% during that year. Because 2.1% of $50.00$ is $1.05$, in December of 1949 the chair cost $50.00 - 1.05 = \$48.95$.

d. We use the buying power formula (Formula 4.13) with $i = 41\%$:

\[
\text{Percent decrease in buying power} = \frac{100i}{100 + i} = \frac{100 \times 41}{100 + 41} = \frac{4100}{141} \approx 29.1\%.
\]

or about 29.1%. The buying power of the Ethiopian birr decreased by 29.1%.

e. We want to find the inflation rate for Kenya. We know that the reduction in buying power was 20.3%, so we use the inflation formula (Formula 4.14) with $B = 20.3\%$:

\[
\text{Percent rate of inflation} = \frac{100B}{100 - B} = \frac{100 \times 20.3}{100 - 20.3} = \frac{2030}{79.7} \approx 25.5\%.
\]

or about 25.5%. The inflation rate of 25.5% in Kenya was higher than the inflation rate of 25.0% in Ukraine.
SUMMARY 4.9: INFLATION AND REDUCTION OF CURRENCY BUYING POWER

If the inflation rate is $i$ (expressed as a percent), the change in the buying power of currency can be calculated using

$$\text{Percent decrease in buying power} = \frac{100i}{100 + i}.$$

A companion formula gives the inflation rate in terms of the decrease $B$ (expressed as a percent) in buying power:

$$\text{Percent rate of inflation} = \frac{100B}{100 - B}.$$

Income taxes

Figure 4.9 and Figure 4.10 show tax tables for the year 2000 from the Internal Revenue Service. One shows tax rates for single people, and the other shows tax rates for married couples filing jointly. Note that the tax rates are applied to taxable income. The percentages in the tables are called marginal rates, and they apply only to earnings in excess of a certain amount. With a marginal tax rate of 15%, for example, the tax owed increases by $0.15 for every $1 increase in taxable income. This makes the tax owed a linear function of the taxable income within a given range of incomes. The slope is the marginal tax rate (as a decimal).

Figure 4.9: 2000 tax tables for singles

| If TAXABLE INCOME Is Over But Not Over This Amount Plus This % Of the Excess Over | The TAX Is | |
|---|---|---|---|---|---|---|---|---|
| $0$ | $26.250$ | $0.00$ | $0.00$ | $0.00$ | $0.00$ | $0.00$ | $0.00$ | $0.00$ |
| $26.250$ | $63.550$ | $3,937.50$ | $28\%$ | $26.250$ | $0.00$ | $0.00$ | $0.00$ | $0.00$ |
| $63.550$ | $132.600$ | $14,381.50$ | $31\%$ | $63.550$ | $0.00$ | $0.00$ | $0.00$ | $0.00$ |
| $132.600$ | $288.350$ | $35,787.00$ | $36\%$ | $132.600$ | $0.00$ | $0.00$ | $0.00$ | $0.00$ |
| $288.350$ | -- | $91,857.00$ | $39.6\%$ | $288.350$ | $0.00$ | $0.00$ | $0.00$ | $0.00$ |
SECTION 4.5  Inflation, taxes, and stocks: Managing your money

**Figure 4.10:** 2000 tax tables for married couples filing jointly

| TAXABLE INCOME Is Over But Not Over This Amount | The TAX Is Plus This % Of the Excess Over |
|---|---|---|---|
| $0 | $43,850 | -- | 15% | $0 |
| $43,850 | $105,950 | $6,577.50 | 28% | $43,850 |
| $105,950 | $161,450 | $23,965.50 | 31% | $105,950 |
| $161,450 | $288,350 | $41,170.50 | 36% | $161,450 |
| $288,350 | -- | $86,854.50 | 39.6% | $288,350 |

Note that these marginal rates increase as you earn more and so move from one tax bracket to another. A system of taxation in which the marginal tax rates increase for higher incomes is referred to as a *progressive tax.*

**EXAMPLE 4.34  Calculating the tax: a single person**

In the year 2000 Alex was single and had a taxable income of $70,000. How much tax did she owe?

**Solution:**

According to the tax table in Figure 4.9, Alex owed $14,381.50 plus 31% of the excess taxable income over $63,550. The total tax is

\[
14,381.50 + 0.31 \times (70,000 - 63,550) = 16,381.00.
\]

**TRY IT YOURSELF 4.34**

In the year 2000 Bob was single and had a taxable income of $50,000. How much tax did he owe?

*The answer is provided at the end of this section.*

A person’s taxable income is obtained by subtracting certain *deductions* from total income. Everyone is allowed a lump sum “personal exemption” deduction, but other deductions can be more complicated and may include things like state and

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13In 1996, magazine publisher Steve Forbes ran for president on a platform that included a 17% “flat tax.” In that system everyone would pay 17% of their taxable income no matter what that income is. In a speech he said, “When we’re through with Washington, the initials of the IRS will be RIP.” Forbes did not win the Republican nomination.
local taxes, home mortgage interest, and charitable contributions. Some people do not have very many of these kinds of deductions. In this case they may choose not to itemize them but rather to take what is called a “standard deduction.”

EXAMPLE 4.35  Comparing taxes: the “marriage penalty”

a. In the year 2000 Ann was single and made a salary of $30,000 per year. She did not itemize deductions and instead took the standard deduction of $4400 plus her personal exemption of $2800. What was Ann’s taxable income, and how much income tax did she owe?

b. In the year 2000 Bill was single and made a salary of $30,000 per year. Bill and Ann got married, and they filed jointly. A married couple making $60,000 per year filing jointly in the year 2000 was given a personal exemption of $2800 each plus a standard deduction as a couple of $7350 if they did not itemize. What was Bill and Ann’s taxable income, and how much income tax did they owe if they did not itemize?

c. Explain what you notice about the amount of tax paid by Ann and Bill as separate single people versus the amount they pay as a married couple filing jointly.

Solution:

a. Ann’s deductions came to $2800 + $4400 = $7200. This gives her a total taxable income of $30,000 − $7200 = $22,800. According to the tax tables, Ann owed 15% of $22,800, which is $3420, in income tax.

b. Ann and Bill’s deductions came to 2 × $2800 + $7350 = $12,950. This gave them a total taxable income of $60,000 − $12,950 = $47,050. According to the tax table for married couples, Figure 4.10, they owe $6577.50 plus 28% of the excess taxable income over $43,850. That is

\[ \$6577.50 + 0.28 \times (\$47,050 - \$43,850) = \$7473.50. \]

c. Ann and Bill as a married couple paid $7473.50 in taxes, but if they filed as single people their combined tax liability would be 2 × $3420 = $6840. That is a difference of $633.50. This means that because they were married they had to pay an extra $633.50 in taxes with absolutely no change in their incomes.
The disparity in part c of Example 4.35 is referred to as the “marriage penalty.” Many people believe the marriage penalty is unfair, but there are those who argue that it makes sense to tax a married couple more than two single people. Can you think of some arguments to support the two sides of this question? The impact of the marriage penalty was reduced in 2003. See Exercises 24 through 26.

Claiming deductions lowers your tax by reducing your taxable income. Another way to lower your tax is to take a tax credit. Here is how to apply a tax credit: Calculate the tax owed using the tax tables (making sure first to subtract from the total income any deductions), and then subtract the tax credit from the tax determined by the tables. Because a tax credit is subtracted directly from the tax you owe, a tax credit of $1000 has a much bigger impact on lowering your taxes than a deduction of $1000. That is the point of the following example.

**EXAMPLE 4.36  Comparing deductions and credits: differing effects**

In the year 2000 Betty and Carol were single, and each had a total income of $75,000. Betty took a deduction of $10,000 but had no tax credits. Carol took a deduction of $9000 and had an education tax credit of $1000. Compare the tax owed by Betty and Carol.

**Solution:**

The taxable income of Betty is $75,000 − $10,000 = $65,000. According to the tax table in Figure 4.9, Betty owes $14,381.50 plus 31% of the excess taxable income over $63,550. That tax is

\[
14,381.50 + 0.31 \times (65,000 - 63,550) = 14,831.00.
\]

Betty has no tax credits, so the tax she owes is $14,831.00.

The taxable income of Carol is $75,000 − $9000 = $66,000. According to the tax table in Figure 4.9, before applying tax credits Carol owes $14,381.50 plus 31% of the excess taxable income over $63,550. That tax is

\[
14,381.50 + 0.31 \times (66,000 - 63,550) = 15,141.00.
\]

Carol has a tax credit of $1000, so the tax she owes is

\[
15,141.00 - 1000 = 14,141.00.
\]
Betty owes

\[ 14,831.00 - 14,141.00 = 690.00 \]

more tax than Carol.

**TRY IT YOURSELF 4.36**

In the year 2000 Dave was single and had a total income of $65,000. He took a deduction of $8000 and had a tax credit of $1800. Calculate the tax owed by Dave.

*The answer is provided at the end of this section.*

In Example 4.36 the effect of replacing a $1000 deduction by a $1000 credit was to reduce the tax owed by $690. This is a significant reduction in taxes and highlights the benefits of tax credits.

**The Dow**

In the late nineteenth century, tips and gossip caused stock prices to move because solid information was hard to come by. This prompted Charles H. Dow to introduce the Dow Jones Industrial Average (DJIA) in May, 1896, as a benchmark to gauge the state of the market. The original DJIA was simply the average price of 12 stocks that Mr. Dow picked himself. Today the Dow, as it is often called, consists of 30 “blue-chip” U.S. stocks picked by the editors of the *Wall Street Journal*. For example, in June of 2009 General Motors was removed from the list as it entered bankruptcy protection. Here is the list as of June 8, 2009.

<table>
<thead>
<tr>
<th>The 30 Dow companies as of June 8, 2009:</th>
</tr>
</thead>
<tbody>
<tr>
<td>3M Company • Alcoa Incorporated • American Express Company • AT&amp;T Incorporated • Bank of America Corporation • Boeing Company • Caterpillar Incorporated • Chevron Corporation • Cisco Systems Incorporated • Coca-Cola Company • E.I. DuPont de Nemours &amp; Company • Exxon Mobil Corporation • General Electric Company • Hewlett-Packard Company • Home Depot Incorporated • Intel Corporation • International Business Machines Corporation • Johnson &amp; Johnson • JPMorgan Chase &amp; Company • Kraft Foods Incorporated • McDonald’s Corporation • Merck &amp; Company Incorporated • Microsoft Corporation • Pfizer Incorporated • Procter &amp; Gamble Company • Travelers Companies Incorporated • United Technologies Corporation • Verizon Communications Incorporated • Wal-Mart Stores Incorporated • Walt Disney Company</td>
</tr>
</tbody>
</table>

As we said earlier, the original DJIA was a true average—that is, you simply
added up the stock prices of the 12 companies and divided by 12. In 1928, a divisor of 16.67 was used to adjust for mergers, takeovers, bankruptcies, stock splits, and company substitutions. Today, they add up the 30 stock prices and divide by 0.132319125 (the divisor), or equivalently, multiply by \(1/0.132319125\) or about 7.56. This means that for every $1 move in any Dow company’s stock price, the average changes by about 7.56 points. (The DJIA is usually reported using two decimal places.)

**EXAMPLE 4.37  Finding changes in the Dow: Disney goes up**

Suppose the stock of Walt Disney increases in value by $3 per share. If all other Dow stock prices remain unchanged, how does this affect the DJIA?

**Solution:**

Each $1 increase causes the average to increase by about 7.56 points. So a $3 increase would cause an increase of about \(3 \times 7.56 = 22.68\) points in the Dow.

**TRY IT YOURSELF 4.37**

Suppose the stock of Microsoft decreases in value by $4 per share. If all other Dow stock prices remain unchanged, how does this affect the DJIA?

_The answer is provided at the end of this section._

The graph in Figure 4.11 shows how the Dow has moved over the last several decades. In Exercises 32 through 35 we explore a few of the more common types of stock transactions.

**Figure 4.11: The Dow**
Derivation of the buying power formula

ALGEBRAIC SPOTLIGHT 4.6
Buying power formula

Suppose the inflation rate is \( i \)% per year. We want to derive the buying power formula

\[
\text{Percent decrease in buying power} = \frac{100i}{100+i}.
\]

Suppose we could buy a commodity, say 1 pound of flour, for 1 dollar a year ago. An inflation rate of \( i \)% tells us that today that same pound of flour would cost \( 1 + \frac{i}{100} \) dollars. To find the new buying power of the dollar, we need to know how much flour we could buy today for 1 dollar. Because 1 pound of flour costs \( 1 + \frac{i}{100} \) dollars, 1 dollar will buy

\[
\frac{1}{1 + \frac{i}{100}} = \frac{100}{100+i}
\]
pounds of flour.

This quantity represents a decrease of

\[
1 - \frac{100}{100+i} = \frac{i}{100+i}
\]

from the 1 pound we could buy with 1 dollar a year ago. The percentage decrease in the amount of flour we can buy for 1 dollar is

\[
\text{Percentage decrease} = \frac{\text{Decrease}}{\text{Previous value}} \times 100\% = \frac{\frac{i}{100+i}}{1} \times 100\% = \frac{100i}{100+i}.
\]

This is the desired formula for the percentage decrease in buying power.

Try It Yourself Answers

TRY IT YOURSELF 4.30 Calculating inflation: CPI increase to 205: 4.7%

TRY IT YOURSELF 4.31 Calculating decrease in buying power: 5% inflation: 23.7%

TRY IT YOURSELF 4.32 Calculating inflation: 2.5% decrease in buying power: 5.5%

TRY IT YOURSELF 4.34 Calculating the tax: a single person: $10,587.50

TRY IT YOURSELF 4.36 Comparing deductions and credits: differing effects: $10,747.50

TRY IT YOURSELF 4.37 Finding changes in the Dow: Disney goes up: The DJIA decreases by 30.24 points.
SECTION 4.5 Inflation, taxes, and stocks: Managing your money

Exercise Set 4.5

Figure 4.12 and Figure 4.13 show tax tables for the year 2003 that we will refer to in the exercises. There were some significant changes from 2000 to 2003.

**Figure 4.12: 2003 tax tables for singles**

<table>
<thead>
<tr>
<th>If TAXABLE INCOME</th>
<th>The TAX Is</th>
</tr>
</thead>
<tbody>
<tr>
<td>THEN</td>
<td></td>
</tr>
<tr>
<td>Is Over</td>
<td></td>
</tr>
<tr>
<td>But Not Over</td>
<td></td>
</tr>
<tr>
<td>This Amount</td>
<td></td>
</tr>
<tr>
<td>Plus This %</td>
<td></td>
</tr>
<tr>
<td>Of the Excess</td>
<td></td>
</tr>
<tr>
<td>Over</td>
<td></td>
</tr>
</tbody>
</table>

**SCHEDULE X —**

<table>
<thead>
<tr>
<th>Single</th>
<th>$0</th>
<th>$7,000</th>
<th>$0.00</th>
<th>10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$7,000</td>
<td>$28,400</td>
<td>$700.00</td>
<td>15%</td>
<td>$7,000</td>
</tr>
<tr>
<td>$28,400</td>
<td>$68,800</td>
<td>$3,910.00</td>
<td>25%</td>
<td>$28,400</td>
</tr>
<tr>
<td>$68,800</td>
<td>$143,500</td>
<td>$14,010.00</td>
<td>28%</td>
<td>$68,800</td>
</tr>
<tr>
<td>$143,500</td>
<td>$311,950</td>
<td>$14,350.00</td>
<td>33%</td>
<td>$143,500</td>
</tr>
<tr>
<td>$311,950</td>
<td>--</td>
<td>$90,514.50</td>
<td>35%</td>
<td>$311,950</td>
</tr>
</tbody>
</table>

**Figure 4.13: 2003 tax tables for married couples**

<table>
<thead>
<tr>
<th>If TAXABLE INCOME</th>
<th>The TAX Is</th>
</tr>
</thead>
<tbody>
<tr>
<td>THEN</td>
<td></td>
</tr>
<tr>
<td>Is Over</td>
<td></td>
</tr>
<tr>
<td>But Not Over</td>
<td></td>
</tr>
<tr>
<td>This Amount</td>
<td></td>
</tr>
<tr>
<td>Plus This %</td>
<td></td>
</tr>
<tr>
<td>Of the Excess</td>
<td></td>
</tr>
<tr>
<td>Over</td>
<td></td>
</tr>
</tbody>
</table>

**SCHEDULE Y-1 —**

<table>
<thead>
<tr>
<th>Married Filing</th>
<th>$0</th>
<th>$14,000</th>
<th>$0.00</th>
<th>10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jointly or Qualifying</td>
<td>$14,000</td>
<td>$56,600</td>
<td>$1,400.00</td>
<td>15%</td>
</tr>
<tr>
<td>$56,600</td>
<td>$114,650</td>
<td>$7,820.00</td>
<td>25%</td>
<td>$56,600</td>
</tr>
<tr>
<td>$114,650</td>
<td>$174,700</td>
<td>$22,282.50</td>
<td>28%</td>
<td>$114,650</td>
</tr>
<tr>
<td>$174,700</td>
<td>$311,950</td>
<td>$39,096.50</td>
<td>33%</td>
<td>$174,700</td>
</tr>
<tr>
<td>$311,950</td>
<td>--</td>
<td>$84,389.00</td>
<td>35%</td>
<td>$311,950</td>
</tr>
</tbody>
</table>
1. **Large inflation rate:** The largest annual rate of inflation in the CPI table, Table 4.3 on page 419, was 18.1% for the year 1946. What year saw the next largest rate? Why do you think the inflation rate in the table for 1945 is blank?

**Food inflation:** The following excerpt from an article at the website of the *Press-Enterprise* discusses inflation of food prices.

**Chicken, pork costs to surge with grains**

**FOOD INFLATION:** Shoppers are only beginning to feel the impact of higher food prices, an executive says.

By ELLEN SIMON The Associated Press Monday, 09:01 PM PDT on May 5, 2008.

Americans may be getting another helping of food inflation, and it seems likely to come from higher prices for chicken and pork.

Overall food inflation could double this year, lifted by the rising costs of fuel, corn and soybeans, some analysts predict.

Food inflation hit 4 percent last year, up from 2.4 percent in 2006. While beef prices were already high, chicken and pork prices didn’t reflect record costs for feed and fuel. That’s poised to change as chicken and pig producers who have been losing money slaughter more animals to decrease the supply and raise the prices they can charge.

Exercises 2 through 4 refer to this article.

2. If your food bill was $3000 in 2005, what was it in 2006?

3. If your food bill was $3000 in 2006, what was it in 2007?

4. Assume that the rate of food inflation did double from 2007 to 2008. If your food bill was $3000 in 2007, what was it in 2008?

**Inflation compounded:** In Exercises 5 and 6 we see the cumulative effects of inflation. We refer to Table 4.3 on page 419.

5. Find the three-year inflation rate from December 1977 to December 1980.
6. Consider the one-year inflation rate for each of the three years from December 1977 to December 1980. Find the sum of these three numbers. Is the sum the same as your answer to Exercise 5?

More on compounding inflation: Here is a hypothetical CPI table.

<table>
<thead>
<tr>
<th>Year</th>
<th>Hypothetical CPI</th>
<th>Inflation rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1935</td>
<td>10</td>
<td>—</td>
</tr>
<tr>
<td>1936</td>
<td>20</td>
<td>%</td>
</tr>
<tr>
<td>1937</td>
<td>40</td>
<td>%</td>
</tr>
<tr>
<td>1938</td>
<td>80</td>
<td>%</td>
</tr>
</tbody>
</table>

Use this table for Exercises 7 through 10.

7. Fill in the missing inflation rates.

8. Find the three-year inflation rate from 1935 to 1938.

9. Find the sum of the three inflation rates during the three years from 1935 to 1938. Is your answer here the same as your answer to Exercise 8?

10. Use the idea of compounding from Section 4.1 to explain the observation in Exercise 9.

11. Guinean franc: Table 4.4 on page 420 shows that the inflation rate for Guinea in the year 2008 was 30%. By how much did the buying power of the Guinean franc (the currency in Guinea) decrease during 2008?

12. Find the inflation rate: Suppose the buying power of a dollar went down by 60% over a period of time. What was the inflation rate during that period?

13. Continuing inflation: Suppose that prices increase 3% each year for 10 years. How much will a jacket that costs $80 today cost in 10 years? Suggestion: The price of the jacket increases by the same percentage each year, so the price is an exponential function of the time in years. You can think of the price as the balance in a savings account with an APY of 3% and an initial investment of $80.

14. More on continuing inflation: This is a continuation of Exercise 13. If prices increase 3% each year for 10 years, what is the percentage decrease in the buying power of currency over the 10-year period?
Flat tax: Steve Forbes ran for U.S. President in 1996 and 2000 on a platform proposing a 17% flat tax, that is, that the income tax should simply be 17% of the taxable income. Suppose that Alice was single in the year 2003 with a taxable income of $30,000 and that Joe was single in the year 2003 with a taxable income of $300,000. Use this information for Exercises 15 through 19. For Exercises 15 and 16 use the tax tables on page 431.

15. What was Alice’s tax?

16. What was Joe’s tax?

17. If the 17% flat tax proposed by Mr. Forbes had been in effect in 2003, what would Alice’s tax have been?

18. What would Joe’s tax be under the 17% flat tax?

19. When you compare Alice and Joe, what do you think about the fairness of the flat tax versus a progressive tax?

More on the flat tax: Let’s return to Alice and Joe from Exercises 15 through 19. We learn that Joe actually made $600,000, but his taxable income was only $300,000 because of various deductions allowed by the system in 2003. PropONENTs of the flat tax say that many of these deductions should be eliminated, so the 17% flat tax should be applied to Joe’s entire $600,000. Use this information for Exercises 20 through 23.

20. What would Joe’s tax be under the 17% flat tax?

21. How much more tax would Joe pay than under the 2003 system?

22. How much income would Joe have to make for the 17% flat tax to equal the amount he pays in the year 2003 with a taxable income of $300,000?

23. When you compare Alice and Joe now, what do you think about the fairness of the flat tax versus a progressive tax?
2003 taxes: This exercise re-does Example 4.35 using the tax tables and deduction rates for the year 2003. In 2003 the personal deduction was $3050, the standard deduction of a single person was $4750, and the standard deduction for a couple filing jointly was $9500. Use this information for Exercises 24 through 26.

24. In the year 2003 Ann was single and made a salary of $30,000 per year. She took the standard deduction (plus her personal deduction). What was Ann’s taxable income, and how much income tax did she owe?

25. In the year 2003 Bill was single and made a salary of $30,000 per year also. Bill and Ann got married. They filed jointly and took the standard deduction (plus their personal deductions). What was their taxable income, and how much income tax did they owe?

26. Explain what you notice about the amount of tax paid by Ann and Bill as separate single people versus the amount they pay as a married couple filing jointly.

27. Bracket creep: At the start of 2003 your taxable income was $28,400, and you received a cost-of-living raise because of inflation. Suppose that inflation was 4% and that your raise resulted in a 4% increase in your taxable income. By how much, and by what percent, did your taxes go up over what they would have been without a raise? (Assume that you were single in 2003.) Remark: Note that your buying power remains the same, but you’re paying higher taxes. Not only that, but you’re paying at a higher marginal rate! This phenomenon is known as “bracket creep,” and federal tax tables are adjusted each year to account for this.

28. Deduction and credit: In the year 2003 Ethan was single and had a total income of $55,000. He took a deduction of $9000 and had a tax credit of $1500. Calculate the tax owed by Ethan.

29. Moving DJIA: Suppose the stock of McDonald’s increases in value by $2 per share. If all other Dow stock prices remain unchanged, how does this affect the DJIA?

30. Average price: What is the average price of a share of stock in the Dow list when the DJIA is 10,000?

31. Dow highlights: Use Figure 4.11 to determine in approximately what year the DJIA first reached 5000. About when did it first reach 10,000?
Exercises 32 through 35 are suitable for group work.

Stock market transactions: There are any number of ways to make (or lose) money with stock market transactions. In Exercises 32 through 35 we will explore a few of the more common types of transactions. (Fees for such transactions will be ignored.)

32. **Market order:** The simplest way to buy stock is the *market order*. Through your broker or on-line you ask to buy 100 shares of stock X at market price. As soon as a seller is located, the transaction is completed at the prevailing price. That price will normally be very close to the latest quote, but the prevailing price may be different if the price fluctuates between the time you place the order and the time the transaction is completed. Suppose you place a market order for 100 shares of stock X and the transaction is completed at $44 per share. Two weeks later the stock value is $58 per share and you sell. What is your net profit?

33. **Limit orders:** If you want to insist on a fixed price for a transaction you place a *limit order*. That is, you offer to buy (or sell) stock X at a certain price. If the stock can be purchased for that price, the transaction is completed. If not, no transaction occurs. Often limit orders have a certain expiration date. Suppose you place a limit order to buy 100 shares of Stock X for $40 per share. When the stock purchase is completed, you plan to place immediately a limit order to sell stock X at $52 per share. The following table shows the value of stock X. On which days are these two transactions completed, and what is your profit?

<table>
<thead>
<tr>
<th>Date</th>
<th>Market price</th>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
<th>Day 6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$44</td>
<td>$45</td>
<td>$40</td>
<td>$48</td>
<td>$52</td>
<td>$50</td>
</tr>
</tbody>
</table>
34. **Stop loss and trailing stops:** If you own a stock, a *stop loss* order protects you from large losses. For example, if you own 100 shares of stock X, bought at $45 per share, you might place a stop loss order for $40 per share. This order automatically sells your stock if the price drops to $40. No matter what happens, you can’t lose more than $5 per share. A similar type of order that protects profits is the *trailing stop*. The trailing stop order sells your stock if the value goes below a certain percentage of the market price. If the market price remains the same or drops, the trailing stop doesn’t change and acts like a stop loss order. If on the other hand the market price goes up, the trailing stop follows it so that it protects profits. Suppose for example that you own 100 shares of stock X, that the market price is $40 per share, and that you place a trailing loss order of 5%. If the price drops by $2 (5% of 40 dollars), the stock is sold. If on the other hand the market value increases to $44, then you will sell the stock when it declines by 5% of 44 dollars. That is, you sell when the market price drops to $41.80. Consider the table in Exercise 33, and suppose that you purchase 100 shares of the stock and place a 5% trailing stop order on day 1. On which day (if any) will your stock be sold?

35. **Selling short:** *Selling short* is the selling of stock you do not actually own but promise to deliver. Suppose you place an order to sell short 100 shares of stock X at $35. Eventually your order must be *covered*. That is, you must sell 100 shares of stock X at a value of $35 per share. If when the order is covered the value of the stock is less than $35 per share then you make money; otherwise you lose money. Suppose that on the day you must cover the sell short order the price of stock X is $50 per share. How much money did you lose?

36. **History: More on selling short:** In 1992, George Soros “broke the Bank of England” by selling short the British pound. Write a brief report on his profit and exactly how he managed to make it.

37. **History: The Knights Templar:** The Knights Templar was a monastic order of knights founded in 1112 to protect pilgrims traveling to the Holy Land. Recent popular novels have revived interest in them. Some have characterized the Knights Templar as the first true international bankers. Report on the inter-
national aspects of their early banking activities.

38. **History: The stock market**: The Dow Jones Industrial Average normally fluctuates, but over the last half century it has generally increased. Dramatic drops in the Dow (stock market crashes) can have serious effects on the economy. Report on some of the most famous of these. Be sure to include the crash of 1929.

39. **History: The Federal Reserve Bank**: The Federal Reserve Bank is an independent agency that regulates various aspects of American currency. Write a report on the Federal Reserve Bank. Your report should include the circumstances of its creation.

40. **History: The SEC**: The Securities Exchange Commission regulates stock market trading in the U.S. Write a report on the creation and function of the SEC.

The following exercises are designed to be solved using technology such as calculators or computer spreadsheets. For assistance see the technology supplement.

**Mortgage interest deduction**: Interest paid on a home mortgage is normally tax-deductible. That is, you can subtract the total mortgage interest paid over the year in determining your taxable income. This is one advantage of buying a home. Suppose you take out a 30-year home mortgage for $250,000 at an APR of 8% compounded monthly. Use this information for Exercises 41 through 44.

41. Determine your monthly payment using the monthly payment formula in Section 4.2.

42. Make a spreadsheet that shows for each month of the first year your payment, the amount that represents interest, the amount toward the principal, and the balance owed.

43. Use the results of Exercise 42 to find the total interest paid over the first year. That is what you get to deduct from your taxable income.

44. Suppose that your marginal tax rate is 27%. What is your actual tax savings due to mortgage payments? Does this make the $250,000 home seem a bit less expensive?
This chapter is concerned with financial transactions of two basic kinds: saving and borrowing. We also consider important financial issues related to inflation, taxes, and the stock market.

**Saving money: The power of compounding**

The principal in a savings account typically grows by interest earned. Interest can be credited to a savings account in two ways: as *simple interest* or as *compound interest*. For simple interest the formula for the interest earned is

\[ \text{Simple interest earned} = \text{Principal} \times \text{Yearly interest rate (as a decimal)} \times \text{Time in years}. \]

Financial institutions normally compound interest and report the *annual percentage rate* or APR. The interest rate for a given compounding period is calculated using

\[ \text{Period interest rate} = \frac{\text{APR}}{\text{Number of periods in a year}}. \]

We can calculate the account balance after \( t \) periods using the compound interest formula

\[ \text{Balance after} \ t \ \text{periods} = \text{Principal} \times (1 + r)^t. \]

Here \( r \) is the period interest rate expressed as a decimal, and it should not be rounded. In fact, it is best to do all the calculations and then round.

The *annual percentage yield* or APY is the actual percentage return in a year. It takes into account compounding of interest and is always at least as large as the APR. If \( n \) is the number of compounding periods per year,

\[ \text{APY} = \left(1 + \frac{\text{APR}}{n}\right)^n - 1. \]

Here both the APR and the APY are in decimal form. The APY can be used to calculate the account balance after \( t \) years:

\[ \text{Balance after} \ t \ \text{years} = \text{Principal} \times (1 + \text{APY})^t. \]

Here the APY is in decimal form.
The present value of an investment is the amount we initially invest. The future value is the value of that investment at some specified time in the future. If the investment grows by compounding of interest, these two quantities are related by the compound interest formula. We can rearrange that formula to give the present value we need for a desired future value:

\[
\text{Present value} = \frac{\text{Future value}}{(1 + r)^t}.
\]

In this formula \( t \) is the total number of compounding periods, and \( r \) is the period interest rate expressed as a decimal.

The Rule of 72 can be used to estimate how long it will take for an account growing by compounding of interest to double in size. It says that the doubling time in years can be approximated by dividing 72 by the APR, where the APR is expressed as a percentage, not as a decimal. The exact doubling time can be found using the formula

\[
\text{Number of periods to double} = \frac{\log 2}{\log(1 + r)}.
\]

Here \( r \) is the period interest rate as a decimal.

**Borrowing: How much car can you afford?**

With an installment loan you borrow money for a fixed period of time, called the term of the loan, and you make regular payments (usually monthly) to pay off the loan plus interest in that time. Loans for the purchase of a car or home are usually installment loans.

If you borrow an amount at a monthly interest rate \( r \) (as a decimal) with a term of \( t \) months, the monthly payment is

\[
\text{Monthly payment} = \frac{\text{Amount borrowed} \times r(1 + r)^t}{((1 + r)^t - 1)}.
\]

It is best to do all the calculations and then round. A companion formula tells how much you can borrow for a given monthly payment:

\[
\text{Amount borrowed} = \frac{\text{Monthly payment} \times ((1 + r)^t - 1)}{(r \times (1 + r)^t)}.
\]
For all loans the monthly payment is at least the amount we would pay each month if no interest were charged, which is the amount of the loan divided by the term (in months) of the loan. This number can be used to estimate the monthly payment for a short-term loan if the APR is not large. For all loans the monthly payment is at least as large as the principal times the monthly interest rate as a decimal. This number can be used to estimate the monthly payment for a long-term loan with a moderate or high interest rate.

A record of the repayment of a loan is kept in an amortization table. In the case of buying a home, an important thing for the borrower to know is how much equity she has in the home. The equity is the total amount that has been paid toward the principal, and an amortization table keeps track of this amount.

Some home loans are in the form of an adjustable-rate mortgage or ARM, where the interest rate may vary over the life of the loan. For an ARM the initial rate is often lower than the rate for a comparable fixed-rate mortgage, but rising rates may cause significant increases in the monthly payment.

Saving for the long term: Build that nest egg

Another way to save is to deposit a certain amount into your savings account at the end of each month. If the monthly interest rate is $r$ as a decimal, your balance is given by

$$\text{Balance after } t \text{ deposits} = \frac{\text{Deposit} \times ((1 + r)^t - 1)}{r}.$$  

The ending balance is often called the future value for this savings arrangement. A companion formula gives the amount we need to deposit regularly in order to achieve a goal:

$$\text{Needed deposit} = \frac{\text{Goal} \times r}{((1 + r)^t - 1)}.$$  

Retirees typically draw money from their nest eggs in one of two ways: either as a perpetuity or as an annuity. An annuity reduces the principal over time, but a perpetuity does not. The principal (your nest egg) is often called the present value.
For an annuity with a term of $t$ months we have the formula

$$\text{Monthly annuity yield} = \frac{\text{Nest egg} \times r(1 + r)^t}{(1 + r)^t - 1}.$$ 

In this formula $r$ is the monthly interest rate as a decimal. A companion formula gives the nest egg needed to achieve a desired annuity yield:

$$\text{Nest egg needed} = \frac{\text{Annuity yield goal} \times ((1 + r)^t - 1)}{r(1 + r)^t}.$$ 

**Credit cards: Paying off consumer debt**

Buying a car or a home usually involves a regular monthly payment that is computed as described earlier. But another way of borrowing is by credit card. If the balance is not paid off by the due date, the account is subject to finance charges. A simplified formula for the amount subject to finance charges is

$$\text{Amount subject to finance charges} = \text{Previous balance} - \text{Payments} + \text{Purchases}.$$ 

Suppose we have a balance on our credit card and decide to stop charging. If we make only the minimum payment the balance is given by the exponential formula

$$\text{Balance after } t \text{ minimum payments} = \text{Initial balance} \times ((1 + r)(1 - m))^t.$$ 

In this formula $r$ is the monthly interest rate and $m$ is the minimum monthly payment as a percent of the balance. Both $r$ and $m$ are in decimal form. The product $(1+r)(1-m)$ should not be rounded when the calculation is performed. Because the balance is a decreasing exponential function, the balance decreases very slowly in the long run.

**Inflation, taxes, and stocks: Managing your money**

The *Consumer Price Index* or CPI is a measure of the average price paid by urban consumers in the United States for a “market basket” of goods and services. The *rate of inflation* is measured by the percent change in the CPI over time.

Inflation reflects a decline of the buying power of the consumer’s dollar. Here is a formula that tells how much the buying power of currency decreases for a given inflation rate $i$ (expressed as a percent):

$$\text{Percent decrease in buying power} = \frac{100 \times i}{100 + i}.$$
A key concept for understanding income taxes is the *marginal tax rate*. With a marginal tax rate of 30%, for example, the tax owed increases by $0.30 for every $1 increase in taxable income. Typically those with a substantially higher taxable income have a higher marginal tax rate. To calculate our taxable income we subtract any *deductions* from our total income. Then we can use the tax tables. To calculate the actual tax we owe we subtract any *tax credits* from the tax determined by the tables.

The *Dow Jones Industrial Average* or DJIA is a measure of the value of leading stocks. It is found by adding the prices of 30 “blue-chip” stocks and dividing by a certain number, the *divisor*, to account for mergers, stock splits, and other factors. With the current divisor, for every $1 move in any Dow company’s stock price the average changes by about 7.56 points.

**Chapter Quiz**

1. We invest $2400 in an account that pays simple interest of 8% each year. Find the interest earned after 5 years.

   **Answer:** $960

   *If you had difficulty with this problem see Example 4.1.*

2. Suppose we invest $8000 in a 4-year CD that pays an APR of 5.5%.

   (a) What is the value of the mature CD if interest is compounded annually?
   (b) What is the value of the mature CD if interest is compounded monthly?

   **Answer:** a. $9910.60 b. $9963.60

   *If you had difficulty with this problem see Example 4.3.*

3. We have an account that pays an APR of 9.75%. If interest is compounded quarterly, find the APY. Round your answer as a percentage to two decimal places.

   **Answer:** 10.11%

   *If you had difficulty with this problem see Example 4.4.*
4. How much would you need to invest now in a savings account that pays an APR of 8% compounded monthly in order to have a future value of $6000 in a year and a half?

Answer: $5323.64

If you had difficulty with this problem see Example 4.7.

5. Suppose an account earns an APR of 5.5% compounded monthly. Estimate the doubling time using the Rule of 72, and calculate the exact doubling time. Round your answers to one decimal place.

Answer: Rule of 72: 13.1 years; exact method: 151.6 months (about 12 years and 8 months).

If you had difficulty with this problem see Example 4.8.

6. You need to borrow $6000 to buy a car. The dealer offers an APR of 9.25% to be paid off in monthly installments over \( \frac{5}{2} \) years.

(a) What is your monthly payment?

(b) How much total interest did you pay?

Answer: a. $224.78 b. $743.40

If you had difficulty with this problem see Example 4.11.

7. We can afford to make payments of $125 per month for 2 years for a used motorcycle. We’re offered a loan at an APR of 11%. What price bike should we be shopping for?

Answer: $2681.95

If you had difficulty with this problem see Example 4.10.

8. Suppose we have a savings account earning 6.25% APR. We deposit $15 into the account at the end of each month. What is the account balance after 8 years?

Answer: $1862.16

If you had difficulty with this problem see Example 4.21.
9. Suppose we have a savings account earning 5.5% APR. We need to have $2000 at the end of 7 years. How much should we deposit each month to attain this goal?

**Answer:** $19.57

*If you had difficulty with this problem see Example 4.22.*

10. Suppose we have a nest egg of $400,000 with an APR of 5% compounded monthly. Find the monthly yield for a 10-year annuity.

**Answer:** $4242.62

*If you had difficulty with this problem see Example 4.24.*

11. Suppose your MasterCard calculates finance charges using an APR of 16.5%. Your previous statement showed a balance of $400, toward which you made a payment of $100. You then bought $200 worth of clothes, which you charged to your card. Complete the following table.

<table>
<thead>
<tr>
<th>Previous balance</th>
<th>Payments</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Answer:**

<table>
<thead>
<tr>
<th>Previous balance</th>
<th>Payments</th>
<th>Purchases</th>
<th>Finance charge</th>
<th>New balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 1</td>
<td>$400.00</td>
<td>$100.00</td>
<td>$200.00</td>
<td>$506.88</td>
</tr>
</tbody>
</table>

*If you had difficulty with this problem see Example 4.26.*

12. Suppose your MasterCard calculates finance charges using an APR of 16.5%. Your statement shows a balance of $900, and your minimum monthly payment is 6% of that month’s balance.

(a) What is your balance after a year and a half if you make no more charges and make only the minimum payment?

(b) How long will it take to get your balance under $100?

**Answer:** a. $377.83 b. 46 monthly payments

*If you had difficulty with this problem see Example 4.29.*
13. Suppose the CPI increases this year from 210 to 218. What is the rate of inflation for this year?

   Answer: 3.8%

   If you had difficulty with this problem see Example 4.30.

14. Suppose the rate of inflation last year was 20%. What was the percentage decrease in the buying power of currency over that year?

   Answer: 16.7%

   If you had difficulty with this problem see Example 4.31.