Lagrangian Points

While investigating the three-body gravitational system using the equations of motion of Lagrangian mechanics rather than Newtonian mechanics, Josef Lagrange discovered five stationary solutions of the *circular restricted three-body problem*. This problem is that of two massive objects M_1 and M_2 in circular orbits around their common center of mass CM and the motion of a third body whose mass m is essentially negligible compared to M_1 and M_2; that is,

$$m \ll M_1 \quad \text{and} \quad m \ll M_2 \qquad \textbf{LP-1}$$

This situation is shown in Figure LP-1. The five stationary solutions are points in space where the position of m relative to M_1 and M_2 remains constant. The gravitational force acting on m provides the centripetal force and is given by

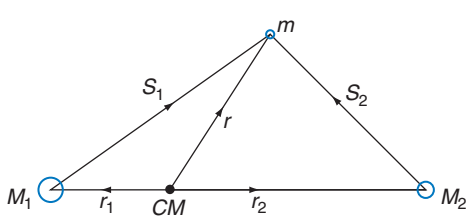
$$\mathbf{F} = -\frac{GM_1 m}{|\mathbf{r} - \mathbf{r_1}|^3}(\mathbf{r} - \mathbf{r_1}) - \frac{GM_2 m}{|\mathbf{r} - \mathbf{r_2}|^3}(\mathbf{r} - \mathbf{r_2}) \qquad \textbf{LP-2}$$

Since both M_1 and M_2 are orbiting their center of mass, $\mathbf{r_1}$ and $\mathbf{r_2}$ are functions of time and the centripetal force on m is then given by

$$\mathbf{F}(t) = m\frac{d\mathbf{r}(t)}{dt} \qquad \textbf{LP-3}$$

The easiest approach to finding the stationary solutions of the equation of motion of the mass m is to transform to a coordinate system whose origin is the center of mass and that rotates with an angular frequency $\boldsymbol{\omega}$ equal to the orbital angular frequency of M_1 and M_2. The magnitude of $\boldsymbol{\omega}$ is given by Kepler's third law: $\omega^2 R^3 = G(M_1 + M_2)$, where $R = |\mathbf{r_1} + \mathbf{r_2}|$ is the separation of M_1 and M_2. In the rotating coordinate system the positions of M_1 and M_2 are fixed and the equation of motion for m now includes the fictitious centrifugal and Coriolis forces. The force $\mathbf{F_m}$ acting on m in the rotating reference frame is then

$$\mathbf{F_m} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\left(\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt}\right) \qquad \textbf{LP-4}$$



LP-1 The circular restricted three-body gravitational potential problem.

where $\mathbf{F}$ is given by Equation LP-2. The second term on the right side of Equation LP-4 is the centrifugal force; the third term is the Coriolis force.

The stationary solutions of Equation LP-4 are those for which $\mathbf{F_m} = 0$. Recalling that the force is the negative gradient of the potential, let us first write down the potential energy U of the mass m, ignoring for the moment the Coriolis force:

$$U = -G\left(\frac{M_1 m}{s_1} + \frac{M_2 m}{s_2}\right) - \frac{1}{2}m\omega^2 r^2 \qquad \textbf{LP-5}$$

where $s_1 = |\mathbf{r} - \mathbf{r_1}|$ and $s_2 = |\mathbf{r} - \mathbf{r_2}|$ as shown in Figure LP-1. The second term on the right side of Equation LP-5 is the centrifugal potential energy. Dividing U by m yields the *effective gravitational potential* (potential energy per unit mass) Φ due to M_1 and M_2:

$$\Phi = \frac{U}{m} = -G\left(\frac{M_1}{s_1} + \frac{M_2}{s_2}\right) - \frac{1}{2}\omega^2 r^2 \qquad \textbf{LP-6}$$

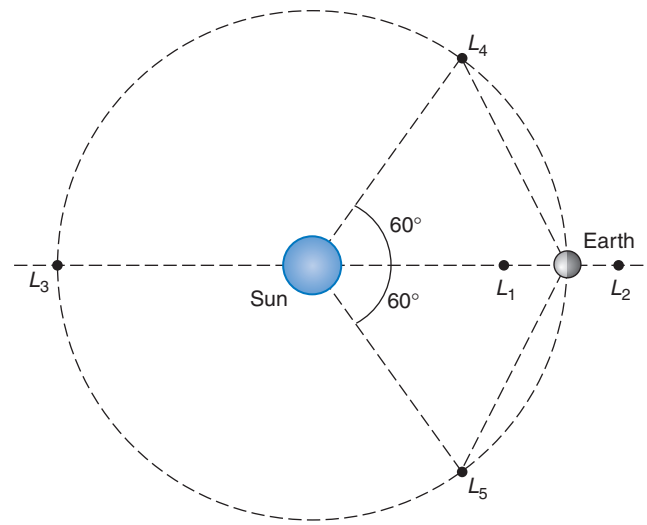
The stationary solutions of the equation of motion for m are those for which

$$\mathbf{F_m} = -\nabla U = -m\nabla\Phi = 0 \qquad \textbf{LP-7}$$

The five stationary solutions found by Lagrange are the points numbered L_1 through L_5 in Figure LP-2, which shows their relative positions. Figure LP-3*a* shows an example of a contour plot of the effective gravitational potential in the *xy* plane (the orbital plane of M_1 and M_2) for the Sun-Earth system. Notice that none of the five appear to be points of stable equilibrium. Points L_1, L_2, and L_3, all on the x axis, are saddle points, that is, the potential decreases from the point in both directions along one axis and increases in both directions along the other axis, just like the surface of a saddle. Figure LP-3*b* illustrates the general shape of this potential along the x axis. A small mass located at any of the three points would, if displaced slightly in the x direction, simply "fall" down the slope, that is, wander away from the point. The time scale of the wandering away is related to the slope at each point: the steeper the slope, the sooner the mass will leave the vicinity of the point and begin chaotic motion.

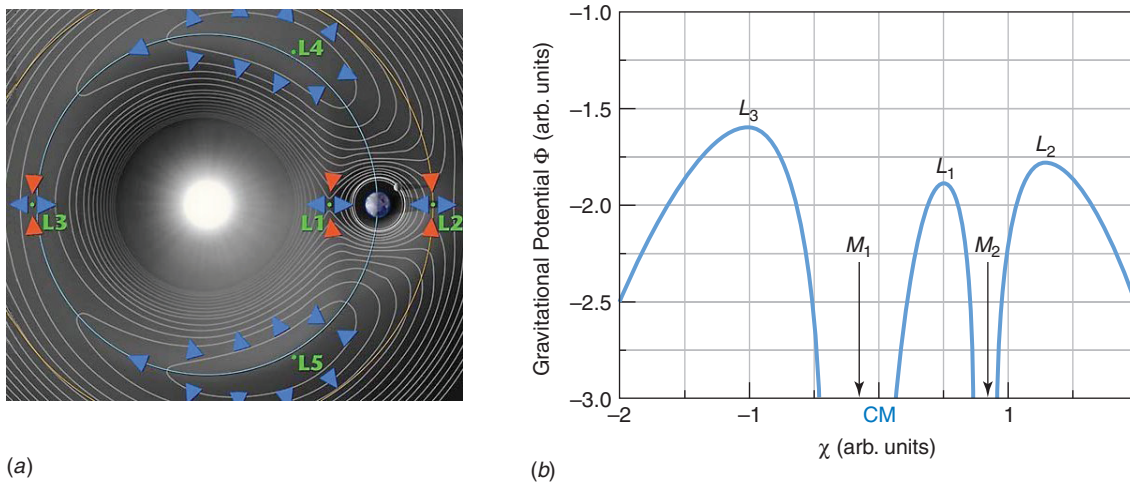
Lagrangian point L_1 is worth some special attention. For L_1 it is intuitively easy to understand why it is an equilibrium point and an unstable one at that. At this point the gravitational attractive forces of M_1 and M_2 on m effectively cancel each other, but if m moves slightly toward one or the other of the large masses, it will continue moving toward that mass and not return to L_1. For a binary star system where one star expands to fill its Roche lobe,[1] mass pushed to L_1 by the expansion pours through the "hole" into the Roche lobe of the second star, leading to potentially catastrophic events for the second star.

The location of L_1 along R depends on the magnitudes of the masses of M_1 and M_2. For the Sun-Earth system, where $M_{\text{Earth}}/M_\odot \approx 3 \times 10^{-6}$, L_1 is located about 0.01 AU $= 1.5 \times 10^6$ km from Earth. Objects located at L_1 orbit the Sun at Earth's angular frequency and have an uninterrupted view of the Sun. This is where the Solar and Heliospheric Observatory Satellite (SOHO) is located. Since L_1 is an unstable equilibrium point, occasional course adjustments are necessary in order to keep the satellite "on station." Typically, this is done by placing SOHO and other satellites located there in small, so-called halo orbits around L_1.



LP-2 The relative positions of the five Lagrangian points for the Sun-Earth system. The diagram is not drawn to scale; points L_1 and L_2 are actually about 0.01 AU from Earth.

[1]The Roche lobe is the volume around the star that is contained within the potential surface that includes the point L_1.



LP-3 (*a*) A contour plot of the gravitational potential on the orbital plane of the Sun-Earth system showing the locations of the five Lagrangian points. The red and blue pointes indicate decreasing and increasing potential, respectively. (*b*) A vertical profile of the gravitational potential along the x axis of a system with $M_1 = 5M_2$. The diagram is not drawn to scale.

At L_2 the gravitational forces acting on m due to M_1 and M_2 are balanced by the centrifugal force. Again using the Sun-Earth system as an example, an object on the x axis orbiting the Sun beyond Earth's location would, in the absence of Earth, have an orbital frequency less than Earth's; however, the presence of Earth's gravitational attraction increases that frequency such that at L_2 the orbital frequency of m equals that of Earth. The location of L_2 is also about 0.01 AU $= 1.5 \times 10^6$ km from Earth on the side away from the Sun. Satellites in the appropriate attitude located at L_2 have an uninterrupted view of deep space. The next-generation space telescope will likely be positioned there. This is where the Microwave Anisotropy Probe (MAP) is located and, like satellites at L_1, it requires periodic course and attitude adjustments in order to remain on station.

Lagrangian point L_3 lies on the x axis beyond the larger of the two masses where the gravitational forces acting on m due to M_1 and M_2 are balanced by the centrifugal force. For the Sun-Earth system, L_3 is located approximately 1 AU from the Sun on the opposite side from Earth slightly outside Earth's orbit. As was described for L_2, an object orbiting the Sun at a distance greater than 1 AU has an orbital frequency less than Earth's. On the x axis on the far side of the Sun the combined effect of the Sun's and Earth's gravity increases the frequency, and at L_3 the orbital frequency of m equals Earth's. L_3 is hidden from Earth at all times. The fictional location of the mysterious Planet X of science-fiction films and books, the instability of L_3 would result in an object at L_3 drifting off into chaotic motion after a few hundred years, so we can be sure that Earth has no sister planet hidden there.

Now let us consider points L_4 and L_5, each located equidistant above and below the x axis at the apex of equilateral triangles with M_1 and M_2 at the other corners. From the potential contours in Figure LP-2*a* we would expect L_4 and L_5 to be points of unstable equilibrium because they sit atop potential hills; however, these points are in fact stable. The reason is that, as a mass m at either point slides down the potential hill, its speed increases and the velocity-dependent Coriolis force in Equation LP-4 becomes effective. The Coriolis force is perpendicular to both $\boldsymbol{\omega}$ and $d\mathbf{r}/dt$ and causes

the object to orbit the Lagrangian point rather than drift away from it provided that the M_1/M_2 ratio meets particular mathematical criteria. These criteria are readily met by the Sun and its planets. For example, the L_4 and L_5 points of Jupiter's orbit are occupied by the three so-called Trojan asteroids Achilles, Agamemnon, and Hector. For this reason L_4 and L_5 are sometimes referred to as the Trojan points. Two of Saturn's moons, Tethys and Dione, each have two small moons at their Trojan points. The Sun-Earth system has no Trojan satellites, but the points do contain clouds of dust orbiting with Earth around the Sun. The L_4 and L_5 points of the Earth-Moon system have been suggested as future locations of extraterrestrial space colonies.