

Liquid-Drop Model and the Semiempirical Mass Formula

The fact that the density and the binding energy per nucleon are approximately the same for all (stable) nuclei was first noticed in the early 1930s, after a sufficient number of atomic masses had been measured. This led to the comparison of the nucleus with a liquid drop, which also has a constant density, independent of the number of molecules. The energy required to remove molecules from a liquid is the heat of vaporization. This is proportional to the mass or number of molecules in the liquid, just as the binding energy is proportional to the number of nucleons. Using this analogy, Weizsäcker in 1935 developed a formula for the mass of a nucleus (or the binding energy, since the two are related by Equation 11-10) as a function of A and Z , called the *Weizsäcker semiempirical mass formula*. We will write down one version of this formula and discuss the origin of the terms. The binding energy is written as

$$B = [+a_1A - a_2A^{2/3} - a_3Z^2A^{-1/3} - a_4(A - 2Z)^2A^{-1} \pm a_5A^{-1/2}] c^2 \quad \mathbf{11-12}$$

The first term in this equation accounts for the fact that the number of interactions is proportional to A and explains why the binding energy per nucleon is approximately constant.

The second term is a correction to the first. The nucleons on the surface of the nucleus have fewer near neighbors, and so fewer interactions, than those in the interior of the nucleus. The effect is analogous to the surface tension of a liquid drop. The surface area is proportional to R^2 , which is proportional to $A^{2/3}$. This term is negative because fewer interactions imply a smaller total binding energy. This is the term that accounts for the sharp decline in the binding energy per nucleon at low A values in Figure 11-10.

The third term accounts for the positive electrostatic energy of a charged drop. Due to the Coulomb repulsion of the protons, this effect equals the average electrostatic energy of a proton-proton pair, about $6ke^2/5R$ (see Problem 11-73) times the number of such pairs, which is $Z(Z - 1)/2$. Thus, the third term is

$$\frac{6}{5} \frac{1}{4\pi\epsilon_0} \frac{e^2}{R} \frac{Z(Z - 1)}{2} \approx \frac{3}{5} \frac{1}{4\pi\epsilon_0} \frac{(Ze)^2}{R_0A^{1/3}} \approx a_3Z^2A^{-1/3} \quad \mathbf{11-13}$$

This positive energy of repulsion decreases the binding energy, so this term is negative. Although this effect exists for all nuclei with $Z > 1$, it is most important for high- Z nuclei and is primarily responsible for the slow decline in the binding energy per nucleon for large values of A .

The fourth term has no analogy in the analysis of a liquid drop. It is a quantum-mechanical term that accounts for the fact that if $N \neq Z$, the energy of the nucleus increases and the binding energy decreases because of the exclusion principle. The quantity $A - 2Z = N + Z - 2Z = N - Z$ is the number of neutrons in excess of the number of protons. The expression $(A - 2Z)^2/A = (N - Z)^2/A$ is an empirical term that is zero if $N = Z$ and is independent of the sign of $N - Z$. It is referred to as the symmetry term.

The last term is an empirical one to account for the pairing tendency of the nucleons that was mentioned earlier in connection with Table 11-2. The contribution to B is positive if Z and N are both even and negative for both Z and N odd. For the case of Z or N even and the other odd, the term is taken to be zero (see Table 11-3). The results of many experiments have been used to fit Equation 11-12, or refinements of it, to the binding energies calculated from the measured masses. The solid curve in Figure 11-10 is one such fit. Table 11-3 lists the values of the coefficients a_1 through a_5 used to produce the curve in Figure 11-10.

From Equations 11-10 and 11-12 and the preceding discussion, Weizsäcker's empirical formula for the mass $M(Z, A)$ of a nucleus can then be written as

$$M(Z, A)c^2 = Zm_p c^2 + Nm_n c^2 - B$$

$$M(Z, A)c^2 = Zm_p c^2 + Nm_n c^2 - [a_1 A - a_2 A^{2/3} - a_3 Z^2 A^{-1/3} - a_4 (A - 2Z)^2 A^{-1} \pm a_5 A^{-1/2}]c^2 \quad \mathbf{11-14}$$

Equation 11-14 is accurate to about ± 0.2 MeV, which is quite good, all things considered. It has many useful applications. For example, a refined version of Equation 11-14 has been used by P. A. Seeger⁹ to compute and tabulate nearly 7500 atomic masses, including many that have obviously not yet been observed. It also provides some helpful panoramic views of nuclear properties. For example, setting $(\partial M/\partial Z)_A = 0$ yields the value of Z for which a series of isobars has minimum mass. Determining the coefficients experimentally with R_0 as a parameter allows its calculation, yielding $R_0 = 1.237$ fm in excellent agreement with the other methods discussed earlier in this chapter. Plotting $M(Z, A)c^2$ values from Equation 11-14 as a third dimension on the N versus Z graph yields a contourlike graph in the rough shape of a valley whose floor lies along the line of stability. The resulting three-dimensional graph is very useful in discussing beta-decay radioactivity, as we will see in Section 11-4.

Table 11-3 "Best-fit" coefficients for the Weizsäcker formula							
Coefficient	a_1	a_2	a_3	a_4	$a_5(N, Z)$		
					Even-even	Odd-odd	Even-odd, odd-even
Value (MeV/ c^2)	15.67	17.23	0.75	93.2	12	-12	0

EXAMPLE 11-4 **The Last Neutron in ${}^4\text{He}$** Find the binding energy of the last neutron in ${}^4\text{He}$.

SOLUTION

1. The binding energy of the second neutron in the ${}^4\text{He}$ nucleus is given by

$$\begin{aligned} B &= (\Delta m)c^2 \\ &= (m({}^3\text{He}) + m_n)c^2 - m({}^4\text{He})c^2 \end{aligned}$$

2. The masses are tabulated in Appendix A:

$$\begin{aligned} m({}^4\text{He}) &= 4.002603 \text{ u} \\ m({}^3\text{He}) &= 3.016030 \text{ u} \\ m_n &= 1.008665 \text{ u} \end{aligned}$$

3. Substituting these values into the expression for B in step 1 gives

$$\begin{aligned} B &= (3.016030 + 1.008665 - 4.002603)c^2 \\ &= (0.022092 \text{ u})c^2 \times \frac{931.5 \text{ MeV}}{(1 \text{ u})c^2} \\ &= 20.58 \text{ MeV} \end{aligned}$$

EXAMPLE 11-5 **Nuclear Mass of ${}^{50}\text{Fe}$** Iron isotopes ${}^{49}\text{Fe}$ and ${}^{51}\text{Fe}$ are both known short-lived radioactive positron emitters, but ${}^{50}\text{Fe}$ has not yet been discovered. Compute the expected value for the nuclear mass of ${}^{50}\text{Fe}$.

SOLUTION

${}^{50}\text{Fe}$ has $Z = 26$, $N = 24$, and $A = 50$. Using the masses of the proton m_p and neutron m_n from Table 11-1 and the values of the Weizsäcker coefficients from Table 11-3, Equation 11-14 yields

$$\begin{aligned} M(26, 50) &= 26 \times 1.007276 \text{ u} + 24 \times 1.008665 \text{ u} \\ &\quad - [15.76 \times 50 - 17.23 \times (50)^{2/3} - 0.75 \times (26)^2 \times (50)^{-1/3} \\ &\quad - 93.2 \times (50 - 2 \times 26)^2 (50)^{-1} + 12 \times (50)^{-1/2}] \text{ MeV}/c^2 \end{aligned}$$

$$M(26, 50) = 50.40 \text{ u} - [410.2] \text{ MeV}/c^2$$

The term in the square brackets is the binding energy. Thus, in energy units the binding energy of ${}^{50}\text{Fe}$ is 410.2 MeV. In unified mass units the binding energy is

$$410.2 \frac{\text{MeV}}{c^2} \frac{1 \text{ u}}{931.5 \text{ MeV}/c^2} = 0.44 \text{ u}$$

The mass of ${}^{50}\text{Fe}$ in u is then

$$M(26, 50) = 50.40 \text{ u} - 0.44 \text{ u} = 49.96 \text{ u}$$

Notice that the binding energy per nucleon for ${}^{50}\text{Fe}$ is about 5.9 MeV, a value well below that of the stable isotopes in the vicinity of $A = 50$ in Figure 11-10.

EXAMPLE 11-6 The Isotopes of Be Beryllium ($Z = 4$) has eight known isotopes, only one of which, ${}^9\text{Be}$, is stable. Compare the atomic mass of ${}^8\text{Be}$ with that of two ${}^4\text{He}$ atoms and the atomic mass of ${}^9\text{Be}$ with that of the combination ${}^7\text{Li}$ and ${}^2\text{H}$. What can be concluded from these comparisons?

SOLUTION

The atomic masses of these isotopes are given in Appendix A as follows:

$$\begin{aligned} {}^2\text{H} &= 2.014102 & {}^8\text{Be} &= 8.005305 & {}^9\text{Be} &= 9.012174 \\ {}^4\text{He} &= 4.002602 & {}^7\text{Li} &= 7.016003 \end{aligned}$$

${}^4\text{He}$ AND ${}^8\text{Be}$

The atomic mass of two ${}^4\text{He}$ atoms is 8.005204 u. The mass of ${}^8\text{Be}$ is larger than that by $1.01 \times 10^{-4} \text{ u} = 0.0941 \text{ MeV}/c^2$. Thus, we would expect the ${}^8\text{Be}$ nucleus to break up into two ${}^4\text{He}$, releasing about 0.0941 MeV in the process. This is indeed what is observed experimentally.

${}^7\text{Li}$ - ${}^2\text{H}$ AND ${}^9\text{Be}$

The sum of the masses of ${}^7\text{Li}$ and ${}^2\text{H}$ is 9.030105 u. The mass of ${}^9\text{Be}$ is smaller by $0.017931 \text{ u} = 16.7 \text{ MeV}/c^2$. This means that the spontaneous disintegration of the ${}^9\text{Be}$ nucleus into a deuteron and ${}^7\text{Li}$ cannot occur.