Production and Sequential Decays

It often occurs that nucleus A decays into another nucleus B, which, in turn, decays into C. For this case, the equation for the decay of nucleus B is not as simple as Equation 11-17 because there must also be a term describing the production of B. If λ_A is the decay constant of A, which decays into B, and λ_B is the decay constant of B, the differential equation giving the decay rate for B is

$$\frac{dN_B}{dt} = -\lambda_B N_B + \lambda_A N_A = -\lambda_B N_B + \lambda_A N_{0A} e^{-\lambda_A t} \qquad \textbf{11-25}$$

The solution of this equation is not too difficult (see, e.g., Problem 11-21) and yields

$$N_B = \frac{N_{0A}\lambda_A}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right) + N_{0B}e^{-\lambda_B t} \qquad \textbf{11-26}$$

Figure 11-13 illustrates a particular case in which $\tau_B = 5\tau_A$. There are also two special cases of interest. The first of these is the case in which nuclide A (called the *parent nuclide*) has a much longer half-life than nuclide B (called the *daughter nuclide*). For this case, $\lambda_A \ll \lambda_B$. For times much less than the half-life of A, the number of A nuclei is approximately constant. Then the rate $\lambda_A N_A \approx \lambda_A N_{0A} = R_0$ and if we start with $N_B = 0$ at time $t = 0$, Equation 11-25 becomes

$$\frac{dN_B}{dt} = R_0 - \lambda_B N_B \qquad \textbf{11-27}$$

which describes the decay when B is produced at a constant rate R_0. For times satisfying $\tau_B \ll t \ll \tau A$, the number of B nuclei is approximately constant and equal to the limiting value

$$N_B = \frac{\lambda_A}{\lambda_B} N_{0A} = \frac{R_0}{\lambda_B} \qquad \textbf{11-28}$$

An example is ^{226}Ra, which decays into ^{222}Rn, which in turn decays into ^{218}Po. The half-life of ^{226}Ra is about 1620 years and the half-life of ^{222}Rn is about 3.83 days. For times greater than about 10 days but much less than 1620 years, the number of ^{222}Rn nuclei remains constant; that is, $dN_B/dt = 0$ because its rate of formation by decay of ^{226}Ra is equal to the rate at which it decays away. This equilibrium situation is called *secular equilibrium*.

In the event that the parent is longer lived than the daughter ($\lambda_A < \lambda_B$), but t is comparable to τ_A, a second case of special interest results, called *transient equilibrium*. In this case after a sufficiently long time has elapsed, dependent on the relative

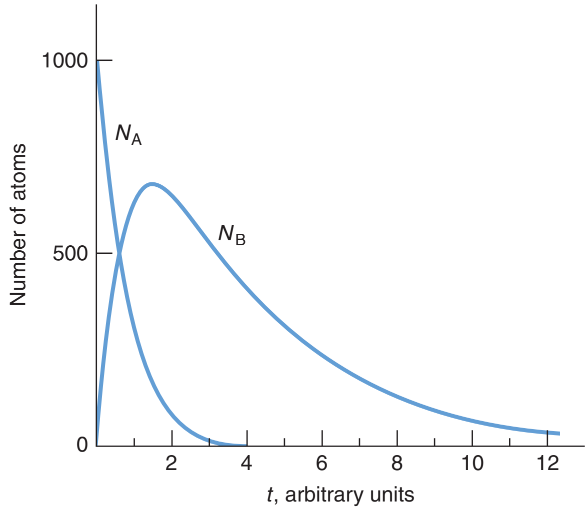

FIGURE 11-13 The number of parent atoms decays according to Equation 11-18 with $N_0 = 1000$. The number of radioactive daughter nuclei varies over time according to Equation 11-26. For the situation illustrated, $\tau_B = 5\tau_A$.

sizes of λ_A and λ_B, $e^{-\lambda_B t}$ becomes negligible in comparison with $e^{-\lambda_A t}$ and, if there are no daughter nuclides present at $t = 0$, Equation 11-26 becomes

$$N_B = \frac{\lambda_A}{\lambda_B - \lambda_A} N_{0A} e^{-\lambda_A t} \tag{11-29}$$

Thus, the daughter eventually decays with the same mean life as the parent.

EXAMPLE 11-9 Determining Production Time In a laboratory experiment, silver foil strips are placed near a neutron source. The capture of neutrons by ^{107}Ag produces ^{108}Ag, which is radioactive and decays by β decay with a half-life of 2.4 min (thus a mean lifetime of about 3.5 min). How long should the foil be left by the neutron source to produce 95 percent of the maximum possible number of radioactive ^{108}Ag nuclei?

SOLUTION

1. The time required to produce N radioactive ^{108}Ag nuclei by bombarding ^{107}Ag at a constant rate R_0 is given by Equation 11-26 with zero ^{108}Ag nuclei at $t = 0$:

$$N = \frac{R_0}{\lambda}(1 - e^{-\lambda t})$$
$$= N_0(1 - e^{-t/\tau})$$

where $N_0 = R_0/\lambda$ is the maximum number possible.

2. If N is to be 95 percent of N_0, then the time required is given by

$$N = 0.95\, N_0 = N_0(1 - e^{-t/\tau})$$

or

$$0.95 = (1 - e^{-t/\tau})$$

3. Rearranging, we have

$$e^{-t/\tau} = 0.05$$

4. Computing τ yields

$$\tau = \frac{t_{1/2}}{0.693} = 3.5 \text{ min}$$

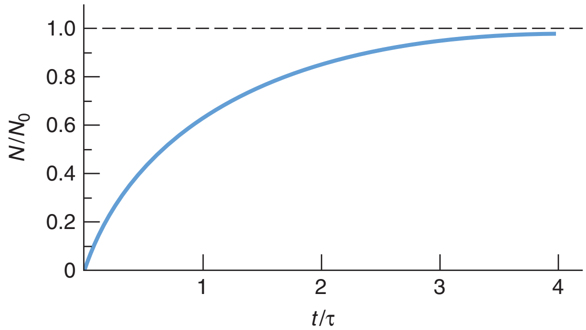

FIGURE 11-14 Growth of number of radioactive nuclei that are produced at a constant rate. The function N/N_0 is given by $N/N_0 = 1 - e^{-t/\tau}$ where τ is the mean life and N_0 is the production rate R_0 times the mean life. N/N_0 is approaching 1 asymtotically as t/τ gets large.

5. Substituting τ into step 3 and solving for t by taking the natural log of both sides gives

$$e^{-t/3.5} = 0.05$$

$$-t/3.5 = -3.00$$

$$t = 10.4 \text{ min}$$

Remarks: *The time required to produce 95 percent of the maximum possible number of* ^{108}*Ag nuclei can also be estimated from Figure 11-14. The estimate from that figure is about three mean lifetimes, in good agreement with our computed value.*

EXAMPLE 11-10 Computing the Activity of a Radioactive Sample The radioactive isotope ^{90}Sr is produced in nuclear reactors and as a result of nuclear weapons testing. Its half-life is 29 years, and its chemical similarity to calcium makes it a potent environmental hazard associated with reactor accidents and weapons tests. It decays to ^{90}Y, which is also radioactive with a half-life of 64 hours. Suppose that we have a sample that consists initially of 1 g of ^{90}Sr. Compute the activity of the sample after 3 days (roughly one half-life of ^{90}Y) and after one month.

SOLUTION

The number of nuclei N_{0A} initially present in the 1 g sample is given by Avogadro's number divided by 90, the molecular weight of ^{90}Sr:

$$N_{0A} = \frac{6.02 \times 10^{23}}{90} = 6.69 \times 10^{21}$$

At $t = 0$, the activity R_{0A} is due entirely to ^{90}Sr and is given by Equation 11-19. The decay constant for ^{90}Sr is

$$\lambda_A = \frac{0.693}{t_{1/2}} = \frac{0.693}{(29 \text{ y})(3.156 \times 10^7 \text{ s/y})} = 7.57 \times 10^{-10} \text{ s}^{-1}$$

Thus, we have

$$R_{0A} = \lambda_A N_{0A} = (7.57 \times 10^{-10} \text{ s}^{-1})(6.69 \times 10^{21})$$

$$R_{0A} = 5.07 \times 10^{12} \text{ decays/s} = 5.07 \times 10^{12} \text{ Bq}$$

AFTER 3 DAYS

After three days have elapsed, the number of ^{90}Y nuclei N_B present is given by Equation 11-26 with $t = (3d)(8.64 \times 10^4 \text{ s/d}) = 2.59 \times 10^5$ s and $N_{0B} = 0$. The ^{90}Y decay constant is

$$\lambda_B = \frac{0.693}{(64\ \text{h})(3600\ \text{s/h})} = 3.01 \times 10^{-6}\ \text{s}^{-1}$$

Substituting into Equation 11-26 yields

$$N_B = (6.69 \times 10^{21})\left[\frac{7.57 \times 10^{-10}}{3.01 \times 10^{-6} - 7.57 \times 10^{-10}}\right] \times \left[e^{-(7.57\times10^{-10})(2.59\times10^{5})} - e^{-(3.01\times10^{-6})(2.59\times10^{5})}\right]$$

Neglecting λ_A in the denominator in comparison with λ_B, we have

$$N_B = (6.69 \times 10^{21})\left[\frac{7.57 \times 10^{-10}}{3.01 \times 10^{-6}}\right](1 - e^{-0.78}) = 9.11 \times 10^{17}$$

and the activity after 3 days is

$$R(3\ \text{d}) = R_A + R_B = R_{0A} + \lambda_B N_B$$

$$R(3\ \text{d}) = 5.07 \times 10^{12} + (3.01 \times 10^{-6})(9.11 \times 10^{17})$$

$$R(3\ \text{d}) = 7.81 \times 10^{12}\ \text{Bq}$$

where we have neglected the change in the number of ^{90}Sr atoms during the three-day period. Notice that the total activity has increased since $t = 0$. About one-third is now due to ^{90}Y.

AFTER 1 MONTH

This time interval is equivalent to more than 11 half-lives of ^{90}Y, so secular equilibrium has been established. The number of ^{90}Y nuclei is now given more conveniently by Equation 11-28:

$$N_B = \left[\frac{7.57 \times 10^{-10}}{3.01 \times 10^{-6}}\right](6.69 \times 10^{21}) = 1.68 \times 10^{18}$$

The total activity after 1 month, again neglecting the change in the number of ^{90}Sr nuclei, is given by

$$R(1\ \text{month}) = R_{0A} + \lambda_B N_B = 5.07 \times 10^{12} + (3.01 \times 10^{-6})(1.68 \times 10^{18})$$

$$R(1\ \text{month}) = 5.07 \times 10^{12} + 5.06 \times 10^{12} = 1.01 \times 10^{13}\ \text{Bq}$$

Notice that the activity of the daughter is equal to that of the parent. This is a consequence of the secular equilibrium.

Questions

4. The radioisotope ^{222}Rn is a relatively short-lived daughter of ^{226}Ra and presents a serious environmental hazard in some regions of the world. Why would it be difficult to remove ^{222}Rn from the environment using ordinary chemical methods?

5. What would be the effect on the activity in a case like Example 11-10 if the result of the daughter nuclide's decay was a radioactive granddaughter with a half-life short compared to that of the daughter?