When Is a Physical Quantity Conserved?

The six conservation laws that we have encountered thus far—energy, charge, angular momentum, linear momentum, baryon number, and lepton number—apply to all four of the fundamental interactions. What is required quantum mechanically for a physical quantity to be conserved?

The time-dependent Schrödinger equation, written in the same form as Equation 6-52, is

$$H_{\text{op}}\Psi = i\hbar\frac{d\Psi}{dt} \qquad \textbf{12-11}$$

where Ψ is the wave function of the system and H_{op} is the Hamiltonian (i.e., energy) operator, assumed here to be independent of time. We saw in Section 6-4 that the expectation value of an observable physical quantity f is given by

$$\langle f \rangle = \int_{-\infty}^{+\infty} \Psi^* f_{\text{op}} \Psi\, dx \qquad \textbf{12-12}$$

where f_{op} is the operator representing the quantity f. For f to be conserved, its value $\langle f \rangle$ must not change in time, so the question is, When is $\langle f \rangle$ independent of time? To answer that question, we assume that f_{op} is independent of time and compute $d\langle f \rangle / dt$ as follows:

$$\frac{d\langle f \rangle}{dt} = \frac{d}{dt}\int \Psi^* f_{\text{op}} \Psi\, dx = \int \frac{d\Psi^*}{dt} f_{\text{op}} \Psi\, dx + \int \Psi^* f_{\text{op}} \frac{d\Psi}{dt}\, dx \qquad \textbf{12-13}$$

The complex conjugate Schrödinger equation is

$$(H_{\text{op}}\Psi)^* = -i\hbar\frac{d\Psi^*}{dt} \qquad \textbf{12-14}\boldsymbol{a}$$

Using the fact that H_{op} is real,[12] this can be written

$$\Psi^* H_{\text{op}} = (H_{\text{op}}\Psi)^* = -i\hbar\frac{d\Psi^*}{dt} \qquad \textbf{12-14}\boldsymbol{b}$$

Combining Equations 12-11 and 12-14*b* with 12-13 results in

$$\frac{d\langle f \rangle}{dt} = \frac{i}{\hbar}\int \Psi^* (H_{\text{op}} f_{\text{op}} - f_{\text{op}} H_{\text{op}})\Psi\, dx \qquad \textbf{12-15}$$

The quantity in parentheses is called the *commutator* of H_{op} and f_{op}. We see from Equation 12-15 that $d\langle f\rangle/dt$ will be zero, that is, f will be conserved, if $(H_{op}f_{op} - f_{op}H_{op}) = 0$. This occurs if $H_{op}f_{op} = f_{op}H_{op}$, in which case we say that H_{op} and f_{op} commute. Thus, we can state the following rule:

> Operators that commute with the Hamiltonian represent conserved physical quantities.

As an obvious example, the Hamiltonian (total energy) operator certainly commutes with itself; therefore, the total energy is a conserved quantity.

Finding such operators is the hard part since the complete form of H_{op} is not often known in nuclear and particle physics. As it turns out, conserved quantities can still be found if it can be shown that H_{op} is invariant under a *symmetry operation* that is related to the physical quantity. As examples, invariance of H_{op} under translation in space leads to conservation of linear momentum, and invariance of H_{op} under translation in time leads to conservation of total energy. An acceptable symmetry operator U_{op} is one that transforms Ψ and Ψ' according to

$$U_{op}\Psi(x,t) = \Psi'(x,t) \tag{12-16}$$

in such a way that the wave function remains normalized and the new Ψ' satisfies the Schrödinger equation. Determining the form of the connection between U_{op} and f_{op} in which f_{op} corresponds to an observable physical quantity is beyond the level of our discussion, but the result, given initially by H. Weyl, is

$$U_{op} = e^{ibf_{op}} \tag{12-17}$$

where b is an arbitrary real quantity independent of x and t. A transformation of this form is called a *global gauge transformation*, where *global* means "everywhere" and *gauge* means "scale." Thus, such a transformation changes the measuring scale everywhere at once. If Ψ' also satisfies the Schrödinger equation, as we stated above, then the Schrödinger equation is *gauge invariant* under the particular symmetry transformationU_{op}; that is, U_{op} has no effect other than changing the scale everywhere. As a consequence, f_{op} represents a conserved quantity. The following example is an illustration of how this works.

EXAMPLE 12-7 Conservation of Charge Use a global gauge transformation to show that electric charge is conserved.

SOLUTION

$\Psi(x,t)$ describes a system with charge q that satisfies Equation 12-11. If we define the charge operator Q_{op}, the $\langle Q\rangle$ will be conserved if H_{op} and Q_{op} commute, that is, if $H_{op}\,Q_{op} = Q_{op}\,H_{op}$. Then $Q_{op}\,\Psi = q\Psi$ and charge q will be conserved.

To see that global gauge invariance ensures that H_{op} and Q_{op} commute, we write

$$\Psi'(x,t) = e^{ibQ_{op}}\Psi(x,t) \tag{12-18}$$

where Ψ' also satisfies Equation 12-11, which becomes

$$H_{op}e^{ibQ_{op}}\Psi = i\hbar\frac{\partial\left(e^{ibQ_{op}}\Psi\right)}{\partial t} \tag{12-19}$$

Multiplying Equation 12-19 by $e^{ibQ_{\text{op}}}$ and noting that Q_{op} is independent of time yields

$$e^{-ibQ_{\text{op}}}H_{\text{op}}e^{ibQ_{\text{op}}}\Psi = i\hbar\frac{\partial\left(e^{-ibQ_{\text{op}}} \times e^{ibQ_{\text{op}}}\Psi\right)}{\partial t} = i\hbar\frac{\partial\Psi}{\partial t} \qquad \textbf{12-20}$$

Comparing Equation 12-20 with Equation 12-11, we see that

$$e^{-ibQ_{\text{op}}}H_{\text{op}}e^{ibQ_{\text{op}}} = H_{\text{op}} \qquad \textbf{12-21}$$

Since b is arbitrary, we select it small enough so that $bQ_{\text{op}} \ll 1$ and expand the exponentials in Equation 12-21 in powers of the exponents, keeping only the first terms, to obtain

$$(1 - ibQ_{\text{op}})H_{\text{op}}(1 + ibQ_{\text{op}}) = H_{\text{op}} \qquad \textbf{12-22}$$

Multiplying this out and discarding the second-order term in bQ_{op}, we have

$$H_{\text{op}}Q_{\text{op}} - Q_{\text{op}}H_{\text{op}} = 0$$

Therefore, $\langle Q \rangle = q$ is conserved. Thus, global gauge invariance ensures the conservation of electric charge.

There are also *local gauge transformations*, where the quantity b in Equation 12-17 is a function of position and time. Although these are mathematically beyond the scope of our discussions, the symmetry of the basic interactions under a number of local gauge transformations leads to several additional conserved quantities. These are discussed in the subsection “More Conservation Laws” in the textbook.