Schrödinger’s Trick

The time-dependent Schrödinger equation for the harmonic oscillator is

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} Kx^2 \psi = i\hbar \frac{\partial \psi}{\partial t} \tag{1}$$

whose stationary, bound-state solutions are

$$\Psi(x,t) = \psi(x) e^{-iEt/\hbar}$$

where \( \psi(x) \) satisfies the time-independent equation

$$\frac{-\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + \frac{1}{2} Kx^2 \psi(x) = E \psi(x) \tag{2}$$

It is not obvious how to solve Equation 2 for the allowed values of \( E \) and the corresponding wave functions \( \psi(x) \). There are several general techniques for solving differential equations; however, this problem can be solved (exactly!) using a beautiful trick invented by Schrödinger.

Recalling that \( \omega = \sqrt{K/m} \), we define \( y = \sqrt{m\omega/\hbar} \) and, correspondingly, \( dy = \sqrt{m\omega/\hbar} \) \( dx \). Note that \( \omega \) is the classical oscillator’s angular frequency: \( x = x_0 \cos \omega t \), which satisfies \( m \left( \frac{d^2 x}{dt^2} \right) = -Kx \). Therefore, substituting \( x \) and \( dx \) in terms of \( y \) and \( dy \) from above into Equation 2, we obtain

$$\frac{-\hbar^2}{2m} \frac{1}{\sqrt{m\omega/\hbar}} \frac{d^2 \psi}{dy^2} + \frac{1}{2} (m\omega^2) \left( \frac{\hbar}{m\omega} \right)^2 y^2 \psi = E \psi$$

and

$$\frac{d^2 \psi}{dy^2} - y^2 \psi = - \frac{2E}{\hbar\omega} \psi \quad \text{or} \quad \left[ \frac{d^2}{dy^2} - y^2 \right] \psi = - \frac{2E}{\hbar\omega} \psi \tag{3}$$

This can be written as

$$\left[ \left( \frac{d}{dy} - y \right) \left( \frac{d}{dy} + y \right) - 1 \right] \psi = - \frac{2E}{\hbar\omega} \psi \tag{4}$$

To see that this is true, note that

$$\left( \frac{d}{dy} - y \right) \left( \frac{d}{dy} + y \right) \psi - \psi = \left( \frac{d}{dy} - y \right) \left( \frac{d\psi}{dy} + y\psi \right) - \psi$$

$$= \frac{d^2 \psi}{dy^2} - y \frac{d\psi}{dy} + y \frac{d\psi}{dy} + y - y^2 \psi - \psi = \frac{d^2 \psi}{dy^2} - y^2 \psi$$
So the Schrödinger equation for the harmonic oscillator becomes

$$\left( \frac{d}{dy} - y \right) \left( \frac{d}{dy} + y \right) \psi = \left( 1 - \frac{2E}{\hbar \omega} \right) \psi$$  \[5\]

Operating on Equation 5 from the left with \( \left( \frac{d}{dy} + y \right) \), we obtain

$$\left( \frac{d}{dy} + y \right) \left( \frac{d}{dy} - y \right) \left( \frac{d}{dy} + y \right) \psi = \left( 1 - \frac{2E}{\hbar \omega} \right) \left( \frac{d}{dy} + y \right) \psi$$

But, for any function \( f \)

$$\left( \frac{d}{dy} - y \right) \left( \frac{d}{dy} + y \right) f = \left( \frac{d}{dy} - y \right) \left( \frac{df}{dy} + yf \right) = \frac{d^2f}{dy^2} + y \frac{df}{dy} - y \frac{df}{dy} - f - y^2 f = \left( \frac{d^2}{dy^2} - y^2 - 1 \right) f$$

This is true for any function \( f(y) \), in particular for \( f(y) = \left( \frac{d}{dy} + y \right) \psi \). Therefore,

$$\left( \frac{d^2}{dy^2} - y^2 \right) \left( \frac{d}{dy} + y \right) \psi - \left( \frac{d}{dy} + y \right) \psi = \left( 1 - \frac{2E}{\hbar \omega} \right) \left( \frac{d}{dy} + y \right) \psi$$

Rearranging this gives us

$$\left( \frac{d^2}{dy^2} - y^2 \right) \left[ \left( \frac{d}{dy} + y \right) \psi \right] = - \frac{2(E - \hbar \omega)}{\hbar \omega} \left[ \left( \frac{d}{dy} + y \right) \psi \right]$$  \[6\]

But recalling Equation 3, which is

$$\left[ \frac{d^2}{dy^2} - y^2 \right] \psi = - \frac{2E}{\hbar \omega} \psi$$

we see that, if we define \( \psi' = \left( \frac{d}{dy} + y \right) \psi \) and \( E' = E - \hbar \omega \), then Equation 6 becomes Equation 7:

$$\left[ \frac{d^2}{dy^2} - y^2 \right] \psi' = - \frac{2E'}{\hbar \omega} \psi'$$  \[7\]

Thus, Equations 3 and 7 have the exact same form. This means that if we have found a solution \( \psi(y) \) corresponding to energy \( E \), then \( \left( \frac{d}{dy} + y \right) \psi = \left( \frac{d\psi}{dy} \right) + y \psi \) is also a solution, and its corresponding energy will be \( E - \hbar \omega \). We can just keep going like this and each time the energy is lowered by \( \hbar \omega \). This means that the spacing of the energy levels of the quantum harmonic oscillator is \( \hbar \omega \).