ERROR BOUNDS FOR NUMERICAL INTEGRATION

In Section 8.1 (ET Section 7.1), we studied three methods of numerical integration: the Trapezoidal Rule T_N, the Midpoint Rule M_N, and Simpson's Rule S_N. These "rules" provide numerical approximations to a definite integral

$$\int_a^b f(x)\,dx$$

For each rule, we stated an Error Bound which provides an upper limit to the size of the error in the approximation. In this supplement, we prove the Error Bounds for T_N and M_N. The Error Bound for S_N may be proved in a similar fashion, but the details are somewhat more complicated and the proof is omitted.

Error Bound for the Midpoint Rule

We treat the Midpoint Rule first. Let $N \geq 1$ be a positive integer. Recall from Section 8.1 (ET Section 7.1) that M_N is equal to the sum of the (signed) areas of the N *midpoint rectangles* (Figure 1). More precisely, we divide the interval $[a, b]$ into N subintervals of length

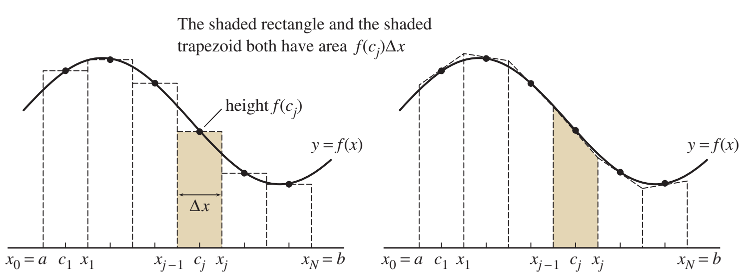
$$\Delta x = \frac{b-a}{N}$$

The endpoints of the subintervals are

$$x_j = a + j\Delta x \qquad\qquad j = 0, 1, \ldots, N$$

and the midpoint of the jth interval $[x_{j-1}, x_j]$ is

$$c_j = a + \left(j - \frac{1}{2}\right)\Delta x$$



(A) M_N is the sum of the areas of the midpoint rectangles.

(B) M_N is also equal to the sum of the areas of the tangential trapezoids.

FIGURE 1

The jth midpoint rectangle is the rectangle of height $f(c_1)$ over the subinterval $[x_{j-1}, x_j]$. This rectangle has signed area

$$f(c_j)\Delta x$$

and M_N is equal to the sum of the signed areas of these rectangles:

$$M_N = \Delta x\big(f(c_1) + f(c_2) + \cdots + f(c_N)\big) \qquad \boxed{1}$$

The error in the approximation M_N is

$$\text{Error}(M_N) = \left|\int_a^b f(x)\,dx - M_N\right|$$

Our goal is to prove the following theorem.

THEOREM 1 Error Bound for M_N Let K_2 be a number such that $|f''(x)| \le K_2$ for all $x \in [a, b]$. Then

$$\text{Error}(M_N) \le \frac{K_2(b-a)^3}{24N^2}$$

Our proof of Theorem 1 uses the interpretation of M_N in terms of *tangential trapezoids*. Recall from Section 8.1 (ET Section 7.1) that M_N is equal to the sum of the signed areas of the trapezoids whose top edges are tangent to the graph of $f(x)$ at the midpoints of the subintervals (Figure 1(B)).The total error Error(M_N) is not greater than the sum of the errors over the subintervals $[x_{j-1}, x_j]$ (it may be less due to cancellation if the tangent line lies above the graph on some intervals and below it others). Therefore, if we denote the error over $[x_{j-1}, x_j]$ by E_j, then

$$\text{Error}(M_N) \le \sum_{j=1}^{N} E_j$$

We will prove that E_j is at most $K_2(\Delta x)^3/24$ for all j, where $\Delta x = (b-a)/N$. In other words,

$$E_j \le \frac{K_2}{24}(\Delta x)^3 = \frac{K_2}{24}\left(\frac{b-a}{N}\right)^3 = \frac{K_2(b-a)^3}{24N^3} \qquad \boxed{2}$$

Since there are N subintervals, the total error is at most N times this quantity:

$$\text{Error}(M_N) \le N\left(\frac{K_2(b-a)^3}{24N^3}\right) = \frac{K_2(b-a)^3}{24N^2}$$

This is the bound stated in Theorem 1. Therefore, Theorem 1 follows if we prove (2).

The error E_j is equal to the signed area between the tangent line and the graph over $[x_{j-1}, x_j]$ (Figure 2). More precisely, let $L(x)$ be the linear approximation to $f(x)$ at $x = c_j$

$$L(x) = f(c_j) + f'(c_j)(x - c_j)$$

The graph of $L(x)$ is the tangent line to the graph at $x = c_j$ and

$$E_j = \left|\int_{x_{j-1}}^{x_j} \big(f(x) - L(x)\big)\,dx\right|$$

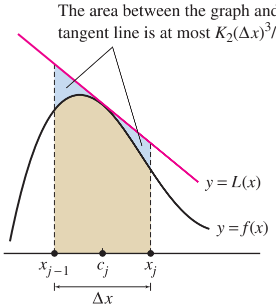


FIGURE 2 The graph of $y = L(x)$ is the tangent line at the midpoint c_j.

The following inequality is valid for all integrable functions $g(x)$:

$$\left|\int_a^b g(x)\,dx\right| \le \int_a^b |g(x)|\,dx \qquad \boxed{3}$$

Using inequality (3), we obtain

$$E_j = \left|\int_{x_{j-1}}^{x_j} \big(f(x) - L(x)\big)\,dx\right| \le \int_{x_{j-1}}^{x_j} \big|f(x) - L(x)\big|\,dx$$

Thus (2) follows from the next theorem.

THEOREM 2 Theorem Assume that $f''(x)$ exists and is continuous. If $|f''(x)| \le K_2$ for all $x \in [x_{j-1}, x_j]$, then

$$\int_{x_{j-1}}^{x_j} \big|f(x) - L(x)\big|\,dx \le \frac{K_2}{24}(\Delta x)^3 \qquad \boxed{4}$$

Proof The linear function $L(x)$ is the first Taylor polynomial for $f(x)$ centered at $x = c_j$. According to the Error Bound for Taylor polynomials (LT Section 9.4, ET Section 8.4),

$$\big|f(x) - L(x)\big| \le \frac{1}{2}K_2(x - c_j)^2$$

Therefore,

$$\int_{x_{j-1}}^{x_j} \big|f(x) - L(x)\big|\,dx \le \frac{1}{2}K_2 \int_{x_{j-1}}^{x_j} (x - c_j)^2\,dx \qquad \boxed{5}$$

This last integral may be computed directly:

$$\int_{x_{j-1}}^{x_j} (x - c_j)^2\,dx = \frac{1}{3}(x - c_j)^3\bigg|_{x_{j-1}}^{x_j} = \frac{1}{3}(x_j - c_j)^3 - \frac{1}{3}(x_{j-1} - c_j)^3$$

Note that

$$x_j - c_j = x_j - \frac{1}{2}(x_{j-1} + x_j) = \frac{1}{2}(x_j - x_j) = \frac{1}{2}\Delta x$$

and similarly, $x_{j-1} - c_j = -\frac{1}{2}\Delta x$. Therefore

$$\int_{x_{j-1}}^{x_j} (x - c_j)^2\,dx = \frac{1}{3}\left(\frac{\Delta x}{2}\right)^3 - \frac{1}{3}\left(\frac{-\Delta x}{2}\right)^3 = \frac{(\Delta x)^3}{12} \qquad \boxed{6}$$

Using (6) in (5), we obtain the desired inequality:

$$\int_a^b \big|f(x) - L(x)\big|\,dx \le \frac{1}{2}K_2\left(\frac{(\Delta x)^3}{12}\right) = \frac{K_2(\Delta x)^3}{24} \qquad \blacksquare$$

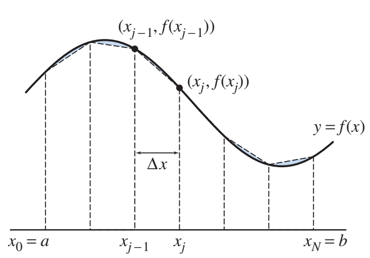


FIGURE 3

Error Bound for the Trapezoidal Rule

The Nth trapezoidal approximation T_N is equal to the sum of the signed areas of the trapezoid obtained by joining the points on the graph above the endpoints $x_0, x_1, \ldots, x_N$. The formula for T_N is

$$T_N = \frac{1}{2}\Delta x\,\big(f(x_0) + 2f(x_1) + \cdots + 2f(x_{N-1}) + f(x_N)\big)$$

In Theorem 3, the error in T_N is defined by

$$\text{Error}(T_N) = \left| \int_a^b f(x)\,dx - T_N \right|$$

THEOREM 3 Error Bound for T_N Let K_2 be a number such that $|f''(x)| \le K_2$ for all $x \in [a, b]$. Then

$$\text{Error}(T_N) \le \frac{K_2(b-a)^3}{12N^2}$$

As we observed above in the case of the Midpoint Rule, the total error $\text{Error}(T_N)$ is not greater than the sum of the errors over the subintervals $[x_{j-1}, x_j]$. Again, we denote the error over $[x_{j-1}, x_j]$ by E_j. Then

$$\text{Error}(T_N) \le \sum_{j=1}^{N} E_j$$

We will prove that E_j is at most $K_2(\Delta x)^3/12$ for all j, where $\Delta x = (b-a)/N$. In other words,

$$E_j \le \frac{K_2}{12}(\Delta x)^3 = \frac{K_2}{24}\left(\frac{b-a}{N}\right)^3 = \frac{K_2(b-a)^3}{12N^3} \qquad \boxed{7}$$

Since there are N subintervals, the total error is at most N times this quantity:

$$\text{Error}(T_N) \le N\left(\frac{K_2(b-a)^3}{12N^3}\right) = \frac{K_2(b-a)^3}{12N^2}$$

This is the bound stated in Theorem 1. Therefore, Theorem 3 follows if we prove (7).

To prove (7), note that the error E_j is equal to the signed area between the secant line and the graph over $[x_{j-1}, x_j]$ (Figure 4). Let $S(x)$ be the linear function whose graph is this secant line. For the record,

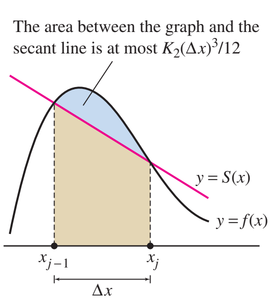


FIGURE 4 The error in the Trapezoidal Rule over $[x_{j-1}, x_j]$ is equal to the area of the blue region between the graph of $f(x)$ and the secant line.

$$S(x) = \left(\frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}}\right)x + \left(\frac{x_j f(x_{j-1}) - x_{j-1} f(x_j)}{x_j - x_{j-1}}\right)$$

However, this formula is not used in the proof. We then have

$$E_j = \left| \int_{x_{j-1}}^{x_j} \big(f(x) - S(x)\big)\,dx \right| \le \int_{x_{j-1}}^{x_j} \big|f(x) - S(x)\big|\,dx$$

The bound (7) follows from the next theorem.

THEOREM 4 Theorem Assume that $f''(x)$ exists and is continuous. If $|f''(x)| \le K_2$ for all $[x_{j-1}, x_j]$,

$$\int_{x_{j-1}}^{x_j} \big|f(x) - S(x)\big|\,dx \le \frac{K_2(\Delta x)^3}{12} \qquad \boxed{8}$$

A key tool in the proof of Theorem 4 is the following version of Rolle's Theorem.

LEMMA 5 Rolle's Theorem for a Function With Three Zeroes Assume that $G(x)$ is continuous on $[x_{j-1}, x_j]$ and that $G''(x)$ exists on (x_{j-1}, x_j). Let $t \in (x_{j-1}, x_j)$ and assume further that

$$G(x_{j-1}) = G(x_j) = G(t) = 0$$

Then there exists $c \in (x_{j-1}, x_j)$ such that $G''(c) = 0$.

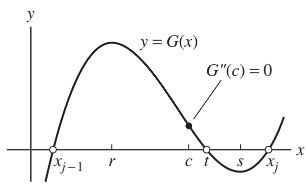


FIGURE 5 The function $G(x)$ has three zeroes at $x = x_{j-1}$, t, and x_j. According to Theorem 5, there exists $c \in (x_{j-1}, x_j)$ such that $G''(c) = 0$.

Proof Rolle's Theorem (Theorem 4 in Section 4.2) states that if $f(x)$ is a continuous function on $[x_{j-1}, x_j]$ such that $f'(x)$ exists on (x_{j-1}, x_j) and

$$f(x_{j-1}) = f(x_j) = 0$$

then there exists $c \in (x_{j-1}, x_j)$ such that $f'(c) = 0$. In other words, *between any two zeroes of $f(x)$ there lies a zero of $f'(x)$*. Applying Rolle's Theorem to $G(x)$ on the interval $[x_{j-1}, t]$, we find that there exists $r \in (x_{j-1}, t)$ such that $G'(r) = 0$. For the same reason, there exists $s \in (t, x_j)$ such that $G'(s) = 0$. Now apply Rolle's Theorem to $G'(x)$ on the interval $[r, s]$. We conclude that there exists $c \in (r, s)$ such that $G''(c) = 0$. ■

We now prove Theorem 4. Let

$$G(x) = f(x) - S(x) - q(x - x_{j-1})(x - x_j)$$

where q is a constant. Given $t \in (x_{j-1}, x_j)$, we may choose q so that $G(t) = 0$. Indeed, we need only solve for q in the equation

$$G(t) = f(t) - S(t) - q(t - x_{j-1})(t - x_j) = 0$$

The solution is

$$q = \frac{f(t) - S(t)}{(t - x_{j-1})(t - x_j)} \qquad \boxed{9}$$

Now observe that

$$G(x_{j-1}) = f(x_{j-1}) - S(x_{j-1}) - q \cdot 0 = 0 - 0 = 0$$

Similarly, $G(x_j) = 0$. Therefore, with our choice of q we find that $G(x)$ has zeroes at $x = x_{j-1}, x_j$ and t. By Lemma 5, there exists $c \in (x_{j-1}, x_j)$ such that $G''(c) = 0$.

We claim that $G''(x) = f''(c) - 2q$. Indeed,

$$\frac{d^2}{dx^2}\big((x - x_{j-1})(x - x_j)\big) = \frac{d^2}{dx^2}\big(x^2 - (x_{j-1} + x_j)x + x_{j-1}x_j\big) = 2$$

On the other hand, $S''(x) = 0$ since $S(x)$ is linear, and thus

$$G''(c) = f''(c) - S''(c) - q\frac{d^2}{dx^2}\big((x - x_{j-1})(x - x_j)\big) = f''(c) - 2q$$

Since $G''(c) = 0$, $q = \frac{1}{2}f''(c)$, and since $|f''(c)| \le K_2$, Eq. (9) gives us

$$|q| = \left|\frac{f(t) - S(t)}{(t - x_{j-1})(t - x_j)}\right| = \frac{1}{2}\left|f''(c)\right| \le \frac{1}{2}K_2$$

Multiplying both sides of this inequality by $|(t - x_{j-1})(t - x_j)|$, we obtain:

$$\boxed{\left|f(t) - S(t)\right| \le \frac{1}{2}K_2\left|(t - x_{j-1})(t - x_j)\right|}$$

This inequality is valid for all $t \in (x_{j-1}, x_j)$. Now write x in place of t in this inequality and integrate:

$$\int_{x_{j-1}}^{x_j} \left|f(x) - S(x)\right| dx \le \frac{1}{2}K_2 \int_{x_{j-1}}^{x_j} \left|(x - x_{j-1})(x - x_j)\right| dx \tag{10}$$

To evaluate this last integral, note that $(x - x_{j-1})(x - x_j)$ is negative for $x \in (x_{j-1}, x_j)$ (since the first factor positive and the second is negative on (x_{j-1}, x_j)). Therefore

$$\int_{x_{j-1}}^{x_j} \left|(x - x_{j-1})(x - x_j)\right| dx = -\int_{x_{j-1}}^{x_j} (x - x_{j-1})(x - x_j)\, dx$$

Direct calculation (which we omit) shows that

$$-\int_{x_{j-1}}^{x_j} (x - x_{j-1})(x - x_j)\, dx = \frac{(x_j - x_{j-1})^3}{6} = \frac{(\Delta x)^3}{6}$$

Using this result in (10) we obtain the desired inequality (8):

$$\int_{x_{j-1}}^{x_j} \left|f(x) - S(x)\right| dx \le \frac{1}{2}K_2 \frac{(\Delta x)^3}{6} = \frac{K_2(\Delta x)^3}{12} \qquad \blacksquare$$